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D-branes in Melvin background

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ABSTRACT: In this paper we discuss D-branes in the Melvin background and its supersymmetric generalizations. In particular we determine the D-brane spectra in these backgrounds by constructing their boundary states explicitly, where some of the D-branes are supersymmetric. The results sensitively depend on whether the value of magnetic flux in the Melvin background is rational or irrational. For the rational case the D-branes are regarded as the generalizations of fractional D-branes in abelian orbifolds $\mathbb{C}^n/\mathbb{Z}_N$ of type-II or type-0 string theory. For the irrational case we found a very limited spectrum. Since the background includes the non-trivial H-flux, the D-branes will provide interesting examples from the viewpoint of the non-commutative geometry.

KEYWORDS: D-branes, Conformal Field Models in String Theory.

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Contents

1. Introduction

One of the important problems in string theory is to understand its vacuum structure completely. Some recent discoveries have revealed the connections between the several vacua whose definitions were originally thought to be different from each other. The first great advance came with the discovery of the string duality [1]. The five string "theories" in flat space turned out to be not different theories but different vacua in a single theory, that is to say, M-theory. The second advance has been made from the investigation of the open string tachyon condensation on unstable and thus non-supersymmetric D-brane systems (for a review see [2, 3]). The key idea of understanding this phenomenon is the Sen's conjecture, which says that the unstable vacuum with D-branes decays to stable lower-dimensional D-branes or the closed string vacuum. From this we can expect that the original vacuum with the unstable D-brane system belongs to the *spontaneous* supersymmetry broken phase. Indeed the non-linear supersymmetries can be realized on such systems [4, 5].

From this we expect that the above observation about open string unstable systems may be applied to closed string unstable vacua such as type-0 string etc. (for the type-0 theory see [6]). If we naively translate the above statement into those examples, such closed string "theories" are not different theories from type II but only different vacua. One attempt in this direction will be the conjecture that type-0A string is equivalent to $\mathbb{S}^1$ compactification of M-theory with the anti-periodic boundary condition for all fermions [7]. Moreover we may speculate that the type-0 string is a spontaneously broken phase of the original thirty two supersymmetries in type II, where the gravitino may obtain the infinite mass and thus disappear. However, in reality it seems difficult to explain it completely within the framework of classical supergravity. For example, one should explain the presence of tachyon in type-0 string and the doubling of RR-fields. These intriguing questions about unstable closed string vacua will also be closely related to the closed string tachyon condensation, which remains to be well understood (for latest discussions see e.g [8]–[14]).

Recently the Kaluza-Klein Melvin backgrounds [15]–[18] have been intensively studied both in M-theory (or superstring with RR flux) [9] and [19]–[28] and in NS-NS superstring model [12, 13, 14, 19, 29, 30]. Both backgrounds will provide useful materials for the studies of unstable closed string vacua since these are known to connect type-0 string with type-II string in the flat space [19]. In the former context the Melvin-like solutions are called flux-branes (Fp-brane) [9, 20, 21, 22]. On the other hand, the latter model gives an exactly solvable superstring model in spite of its non-trivial curved background with H-flux [13, 14, 29, 30]. The simplest model [29, 30] ("9-11" flip of F7-brane) includes three parameters (the radius R and two magnetic fluxes q and β) and thus does cover many vacua. In particular, by tuning these parameters appropriately we can realize the type-0 or type-II string and this shows explicitly that the Melvin background connects type 0 with type II. Furthermore it can be shown that all of the abelian orbifolds $\mathbb{C}/\mathbb{Z}_N$ in type-0 or type-II string are also included as special limits [13]. We can also construct supersymmetric Melvin backgrounds [13, 14] by considering the higher-dimensional generalizations ("9-11" flip of F5-, F3- and F1-brane). These also include supersymmetric orbifolds such as ALE orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ (for ALE space see [31]) as well as non-supersymmetic orbifolds. These Melvin models describe a large region of string vacua both supersymmetric and non supersymmetic, interpolating various orbifolds. Therefore these models will be suitable to explore the structure of vacua in string theory and the decay of unstable vacua.

If we are given a non-trivial background like one of these, a D-brane generally provides a good probe to investigate its stringy geometry. Thus in this paper we will discuss various aspects of D-branes in the Melvin backgrounds. Because the model is exactly solvable as we have mentioned, the boundary conformal field theories of D-branes are also treated exactly. In particular we can construct their boundary states explicitly as we will see later. As a result we find that these D-branes depend on the parameters of the Melvin background in intriguing ways. For example, if we assume the simplest Melvin model [29, 30] and the parameters qR and $\beta\alpha'/R$ are rational numbers, the D-branes which wrap the compactified circle depend only on qR, while those which are point-like along the circles depend only on $\beta\alpha'/R$. This difference sometimes causes the strange Bose-Fermi degeneracy in the open string spectrum on the D-branes in spite of the absence of supersymmetry in the closed string sector. We will later discuss an interpretation of this degeneracy as a remnant of the spontaneously broken supersymmetry.

Even though these two kinds of D-branes are transformed into each other by T-duality, they look different geometrically. The former kind of the D-brane has the spiral world-volume wrapping several times on two-dimensional tori like coils. On the other hand, the latter kind of the D-brane is very similar to fractional D-branes [32, 33] in orbifold theories. Indeed as we have mentioned, if we take appropriate limits, then the model becomes equivalent to orbifolds $\mathbb{C}/\mathbb{Z}_N$ in type-0 and type-II string. We can see that the D-branes in the Melvin background really become the fractional D-branes in the orbifold limit. Especially we find that a D-brane in the Melvin model is divided into an electric fractional D-brane and a magnetic fractional D-brane [7] and [34]–[37] in the type-0 limit. We also find an interesting phenomenon that the type of a fractional-like D-brane is changed due to a kind of monodromy if it goes around the circle.

Moreover, D-brane spectra are very sensitive to whether the magnetic parameters qR and $\beta\alpha'/R$ are rational or irrational. It was found that if we take the decompactified limit in the irrational case, then the closed string sector approaches not the ordinary orbifold $\mathbb{C}/\mathbb{Z}_N$ but an unfamiliar kind of a "large-N limit of the orbifold" [13]. In this case all of the allowed D-branes are pinned at the fixed point of the Melvin background and cannot move around, while in the rational case there exist movable D-branes similar to those which belong to the regular representations in the orbifold theories. Indeed for the irrational parameters if we would try to move the D-branes away which wrap the compactified circle, it should wind infinitely many times ("foliation") and thus it becomes singular.

In this way the D-branes in Melvin backgrounds change their aspects dramatically in accordance with the various values of magnetic parameters.

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It will also be an interesting result that if we consider the higher-dimensional supersymmetric Melvin backgrounds [13, 14], then we can obtain BPS D-branes. In the orbifold limit these D-branes are identified with BPS fractional D-branes on ALE orbifolds $\mathbb{C}^2/\mathbb{Z}_N$.

Finally we would like to mention another interesting viewpoint on the Melvin backgrounds. They include non-trivial H-flux for non-zero β and thus the world-volume theory on the D-branes in these backgrounds may be analyzed in terms of non-commutative geometry [38, 39, 40]. Related examples from this viewpoint may be the D-branes in group manifolds [41, 42], where the non-trivial H-flux is also present. Recently the D-brane charges of these [43] are explained by employing twisted K-theory [44] in [45, 46].

The plan of this paper is as follows. In section 2 we review the Melvin background and the quantization of its sigma model by the operator method. In section 3 we investigate the D-branes in the Melvin background by using the boundary state formalism. We calculate the vacuum amplitude and check the open-closed duality relation. We discuss the geometrical properties of them. We also examine the relation to fractional D-branes in orbifold theories. In section 4 we generalize the arguments in section 3 to those of the higher-dimensional Melvin model. We see that the D-branes in this background become supersymmetric for specific values of the parameters. In section 5 we draw conclusions and discuss the future problems. In appendix we summarize some known facts which are necessary for the analysis in this paper.

2. Closed strings in the Melvin background

In this section we mainly review the exactly solvable model of type-II superstring studied in [29, 30] as well as some recent related developments [7, 13, 19]. See also [47, 48, 49] for such a model in bosonic string theory. We also present the detailed analysis of the world-sheet fermions in NS-R formalism by using superfields.

The target space of this model has the structure of Kaluza-Klein theory and has the topology $\mathrm{M}_3 \times \mathbb{R}^{1,6}$. The three-dimensional manifold M_3 is given by $\mathbb{S}^1$ fibration over $\mathbb{R}^2$. We write the coordinate of $\mathbb{R}^2$ and $\mathbb{S}^1$ by ρ, φ (polar coordinate) and y (with radius R). This non-trivial fibration is due to two Kaluza-Klein (K.K.) gauge fields A_φ and B_φ (see eq. (2.1)) which originate from K.K. reduction of metric $G_{\varphi y}$ and B-field $B_{\varphi y}$, respectively. Thus this background can be viewed as a generalization [16, 17] of the original Melvin solution [15] to string theory, which is recently discussed in the context of the flux brane [9, 14], [20]–[25] and [27, 28]. In the most of the discussions below, we will neglect the trivial flat part $\mathbb{R}^{1,6}$

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The explicit metric and other NSNS fields before the Kaluza-Klein reduction are given as follows

$$ds^2 = d\rho^2 + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)}d\varphi^2 + \frac{1+q^2\rho^2}{1+\beta^2\rho^2}(dy + A_\varphi d\varphi)^2\,, \tag{2.1}$$

$$A_\varphi = \frac{q\rho^2}{1+q^2\rho^2}\,, \qquad B_{\varphi y} \equiv B_\varphi = -\frac{\beta\rho^2}{1+\beta^2\rho^2}\,, \qquad e^{2(\phi-\phi_0)} = \frac{1}{1+\beta^2\rho^2}\,, \tag{2.2}$$

where q, β are the magnetic parameters which are proportional to the strength of two gauge fields and ϕ_0 is the constant value of the dilaton ϕ at $\rho = 0$.

It would be useful to note that if $\beta = 0$, we get a locally flat metric

$$ds^2 = d\rho^2 + dy^2 + \rho^2(d\varphi + q dy)^2\,. \tag{2.3}$$

This background is globally non trivial because the angle φ is compactified such that its period is 2π. For example, its geodesic lines $\varphi + qy =$const. are spiral and do not return to the same point for irrational qR if one goes around the circle $\mathbb{S}^1$. As we will see later, this geometry rules the D-brane spectrum.

2.1 Sigma model description and its free field representation

At first sight the above background (2.2) for general q, β does not seem to be tractable in the description of the two-dimensional sigma model. However, with appropriate T-duality transformations which we will review in appendix A (see also [50]) one can solve this sigma model in terms of free fields [29, 49].[1] In this paper we define the coordinate of world-sheet as $z = \sigma_1 + i\sigma_2$ and the derivatives as $\partial = \frac{1}{2}(\partial_1 - i\partial_2), \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$.

The sigma model for the background (2.2) is given by (we show only the bosonic part)

$$\begin{aligned} S = &\frac{1}{\pi\alpha'}\int d^2\sigma \left[\bar{\partial}\rho\partial\rho + \frac{(1+q^2\rho^2)}{(1+\beta^2\rho^2)}\left(\bar{\partial}Y + \frac{q\rho^2}{1+q^2\rho^2}\bar{\partial}\varphi\right)\left(\partial Y + \frac{q\rho^2}{1+q^2\rho^2}\partial\varphi\right) + \right.\\ &\left. \qquad + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)}\bar{\partial}\varphi\partial\varphi - \frac{\beta\rho^2}{1+\beta^2\rho^2}\left(\bar{\partial}Y\partial\varphi - \partial Y\bar{\partial}\varphi\right)\right]\\ = &\frac{1}{\pi\alpha'}\int d^2\sigma \left[\bar{\partial}\rho\partial\rho + \bar{\partial}Y\partial Y + \right.\\ &\left. \qquad + \frac{\rho^2}{1+\beta^2\rho^2}\left(\bar{\partial}\varphi + (q-\beta)\bar{\partial}Y\right)\left(\partial\varphi + (q+\beta)\partial Y\right)\right]\,, \end{aligned} \tag{2.4}$$

where we have omitted the term of the dilaton coupling for simplicity. We have also abbreviated the fermion terms since it is easily obtained if we use the superfield. They are easily incorporated if we use the $\mathcal{N} = 1$ world-sheet superfield formulation.[2]

[1] In our previous paper [13] we review the path-integral analysis of the Green-Schwarz formalism in the Melvin background, while in this paper we use the operator quantization method of NS-R formalism in order to construct the boundary states.

[2] To do this one has only to replace the derivatives $\partial,\ \bar{\partial}$ with $D_\theta = \partial_\theta + \theta\partial,\ D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}$ and a bosonic field X with $\mathbf{X}(z,\bar{z}) = X(z,\bar{z}) + i\theta\psi_L(z) + i\bar{\theta}\psi_R(\bar{z}) + \cdots$.

First let us perform the T-duality which transforms the field φ into the new one $\tilde{\varphi}$ (for more explanations see the appendix A). The result is given by

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma \left[\bar{\partial}\rho\partial\rho + \left(\bar{\partial}Y + \beta\bar{\partial}\tilde{\varphi}\right)\left(\partial Y + \beta\partial\tilde{\varphi}\right) + \right. \\ \left. + \, q\left(\bar{\partial}\tilde{\varphi}\partial Y - \bar{\partial}Y\partial\tilde{\varphi}\right) + \frac{1}{\rho^2}\bar{\partial}\tilde{\varphi}\partial\tilde{\varphi}\right]. \tag{2.5}$$

After we define the field Y' by

$$Y' = Y + \beta\tilde{\varphi}\,, \tag{2.6}$$

we can again take the T-duality along $\tilde{\varphi}$ into φ'. Then we obtain

$$S = \frac{1}{\pi\alpha'}\int d^2\sigma \left[\bar{\partial}\rho\partial\rho + \bar{\partial}Y'\partial Y' + \rho^2\left(\bar{\partial}\varphi' + q\bar{\partial}Y'\right)\left(\partial\varphi' + q\partial Y'\right)\right]. \tag{2.7}$$

From this expression it is easy to see that one can describe the sigma model by free fields X' and $\bar{X}'$ which are defined by

$$X' = \rho e^{i\varphi''}\,, \qquad \bar{X}' = \rho e^{-i\varphi''}\,, \tag{2.8}$$

where

$$\varphi'' = \varphi' + qY'\,. \tag{2.9}$$

Here we will examine the relation between the free fields $X', \bar{X}'$ and the fields $X = \rho e^{i\varphi}, \bar{X} = \rho e^{-i\varphi}$ which represent the original plane $\mathbb{R}^2 \subset \mathrm{M}_3$ in (2.2). Applying the relation (A.6) to the above two different T-duality transformations,[3] we can obtain

$$\partial\varphi = \partial\varphi'' - q\partial Y - \beta\partial Y'\,, \qquad \bar{\partial}\varphi = \bar{\partial}\varphi'' - q\bar{\partial}Y + \beta\bar{\partial}Y'\,. \tag{2.10}$$

This shows that the field φ is rewritten as

$$\varphi(z,\bar{z}) = \varphi''(z,\bar{z}) - qY(z,\bar{z}) + \beta\left(Y'_R(\bar{z}) - Y'_L(z)\right), \tag{2.11}$$

where $Y'_L(z)$ ($Y'_R(\bar{z})$) is the left(right)-moving part of Y'. Therefore from this the relation between $X', \bar{X}'$ and $X, \bar{X}$ is represented as

$$X(z,\bar{z}) = e^{-iqY(z,\bar{z}) + i\beta Y'_R(\bar{z}) - i\beta Y'_L(z)} X'(z,\bar{z})\,. \tag{2.12}$$

It is easy to generalize the above results into those in the supersymmetric case since we can use $\mathcal{N}=1$ world-sheet superfield formulation. Then the above equation (2.12) does hold as a superfield and we can also define the free fields $(\psi'_{L,R}, \bar{\psi}'_{L,R})$ as the partners of $(X', \bar{X}')$.

[3]The same result can be obtained by performing the T-duality about Y once if we regard $\check{\varphi} \equiv \varphi + qY$ and Y as fundamental fields in the sigma model. In this subsection we have used the T-duality about φ for the convenience of the explanation.

Since we have the free field representation $(Y', \eta'_{L,R})$, $(X', \psi'_{L,R})$ and $(\bar{X}', \bar{\psi}'_{L,R})$, the quantization of the Melvin background can be performed. Before that, we have to examine the boundary condition of the field Y', which is a little subtle analysis in this section. From the relations (A.7) one obtains

$$\begin{aligned} \partial\tilde{\varphi} &= -\rho^2 \partial\varphi'' + i\psi'_L \bar{\psi}'_L \equiv i\alpha' j_L \,, \\ \bar{\partial}\tilde{\varphi} &= \rho^2 \bar{\partial}\varphi'' - i\psi'_R \bar{\psi}'_R \equiv -i\alpha' j_R \,, \end{aligned} \tag{2.13}$$

and the conservation law of the above current $j_{L,R}$ follows directly. Notice also the useful relation

$$\rho^2 \partial\varphi'' = \frac{1}{2i}\left(\bar{X}'\partial X' - X'\partial\bar{X}'\right). \tag{2.14}$$

Then we can define the angular momentum operators $\hat{J}_L, \hat{J}_R$ in $(X', \bar{X}') \in \mathbb{R}^2$ directions as follows[4]

$$\hat{J}_L = \frac{1}{2\pi i}\oint dz\; j_L(z)\,, \qquad \hat{J}_R = -\frac{1}{2\pi i}\oint d\bar{z}\; j_R(\bar{z})\,. \tag{2.15}$$

Then we can see from (2.13) and (2.15) how the boundary condition of $\tilde{\varphi}$ should be twisted

$$\tilde{\varphi}(\tau, \sigma + 2\pi) = \tilde{\varphi}(\tau, \sigma) - 2\pi\alpha' \hat{J}\,, \tag{2.16}$$

where we have defined the total angular momentum operator as $\hat{J} = \hat{J}_R + \hat{J}_L$ and the new world-sheet coordinates τ, σ as $z = \exp(\tau + i\sigma)$. Moreover notice that the original coordinate Y satisfies

$$Y(\tau, \sigma + 2\pi) = Y(\tau, \sigma) + 2\pi R w\,. \tag{2.17}$$

After all from (2.6), (2.16) and (2.17) the periodicity of the field Y' is given by

$$Y'(\tau, \sigma + 2\pi) = Y'(\tau, \sigma) + 2\pi R w - 2\pi\alpha'\beta\hat{J}\,. \tag{2.18}$$

On the other hand, the canonical momentum of Y is

$$\begin{aligned} P_Y &= \frac{1}{2\pi\alpha'}\int d\sigma\,\left(q\bar{\partial}\tilde{\varphi} - q\partial\tilde{\varphi} + \partial Y' + \bar{\partial}Y'\right) \\ &= q\hat{J} + \frac{1}{2}(P'_L + P'_R)\,, \end{aligned} \tag{2.19}$$

[4]Note the operator product expansions (OPE) $j_L(z)\partial X'(w) \sim \frac{1}{2(z-w)}\partial X'(w) + \frac{1}{2(z-w)^2}X'(w)$, $j_L(z)\psi'_L(w) \sim \frac{1}{z-w}\psi'_L(w)$, $j_L(z)\bar{\psi}'_L(w) \sim -\frac{1}{z-w}\bar{\psi}'_L(w)$ and similar results for the right-moving sector. Here we have used the relation (2.13) and (2.14) and OPE for free fields normalized such that $X'(z)\bar{X}'(w) \sim -\alpha' \ln(z-w)$ and $\psi'_L(z)\bar{\psi}'_L(w) \sim \alpha'/(z-w)$. Thus we can find that the operators $\psi'_{L,R}$ and $\partial X', \bar{\partial}X'$ have charges $\hat{J}_{L,R} = 1$ and on the other hand $\bar{\psi}'_{L,R}$ and $\partial\bar{X}', \bar{\partial}\bar{X}'$ have charges $\hat{J}_{L,R} = -1$.

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where the first line is obtained from (2.5) and (2.6). Therefore from the quantization of P_Y as $P_Y = \frac{n}{R}$ ($n \in \mathbb{Z}$) the quantized zero modes of Y' are obtained as follows

$$P'_L + P'_R = 2\left(\frac{n}{R} - q\hat{J}\right), \qquad P'_L - P'_R = 2\left(\frac{Rw}{\alpha'} - \beta\hat{J}\right). \tag{2.20}$$

Next we turn to the quantization of the free fields $X', \bar{X}'$ and $\psi'_{L,R}, \bar{\psi}'_{L,R}$. They obey the following twisted boundary conditions which can be obtained from (2.12), (2.17) and (2.20),

$$X'(\tau, \sigma + 2\pi) = e^{2\pi i\gamma} X'(\tau, \sigma), \tag{2.21}$$
$$\psi'_L(\tau, \sigma + 2\pi) = e^{2\pi i\gamma}\psi'_L(\tau, \sigma), \qquad \psi'_R(\tau, \sigma + 2\pi) = e^{2\pi i\gamma}\psi'_R(\tau, \sigma),$$
$$\text{where} \qquad \gamma \equiv qRw + \beta\alpha'\left(\frac{n}{R} - q\hat{J}\right). \tag{2.22}$$

Note that there are no zero-modes for $X', \bar{X}'$ if γ is not an integer. This fact is crucial when we will consider D-branes in this model later.

The above boundary conditions are similar to those in orbifold theories and therefore it is straightforward to perform the mode expansion and its canonical quantization. We summarize these results in the appendix B.

2.2 Mass spectrum

We have explained that the sigma model of the Melvin background can be solved in terms of free fields $(X', \bar{X}', Y')$. Thus it is straightforward to compute the mass spectrum of this model in the NS-R formalism, which can be obtained from $L_0, \tilde{L}_0$ in (B.4). If we represent it by using $\hat{N}_{L,R}$ in (B.5) and $\hat{J}_{L,R}$ in (B.6), the result [13, 29, 30] is given by

$$\frac{\alpha' M^2}{2} = \frac{\alpha'}{2R^2}\left(n - qR\hat{J}\right)^2 + \frac{R^2}{2\alpha'}\left(w - \frac{\alpha'}{R}\beta\hat{J}\right)^2 +$$
$$+ \hat{N}_R + \hat{N}_L - \hat{\gamma}\left(\hat{J}_R - \hat{J}_L\right), \tag{2.23}$$
$$\text{where} \qquad \hat{\gamma} \equiv \gamma - [\gamma], \tag{2.24}$$

with the level matching constraint

$$\hat{N}_R - \hat{N}_L - nw + [\gamma]\hat{J} = 0, \tag{2.25}$$

where $[\gamma]$ denotes the integer part of γ. Moreover the GSO-projection for type-II theory restricts the above spectrum, which causes a little subtlety [13, 29, 30]. For $2n \leq \gamma < 2n+1$ ($n \in \mathbb{Z}$) it is the standard type-II GSO-projection and the allowed spectra are those which give $\hat{N}_{L,R}$ the integer values for NSNS-sector. However, for $2n+1 \leq \gamma < 2n+2$ it is the *reversed* one, where $\hat{N}_{L,R}$ takes half-integer values for NSNS-sector. This fact can be seen from the one-loop partition function $Z(R, q, \beta)$ [13, 29, 30] of the

Melvin background by comparing the NS-R formalism with the Green-Schwarz formalism, where GSO-projection is not needed. In fact the spectrum (2.23) and (2.25) are the same as those obtained in the Green-Schwarz formalism [13, 29, 30].

Next let us see some interesting symmetries of the partition function $Z(R,q,\beta)$. From the mass spectrum it is easy to show the T-duality relation (see also appendix A)

$$Z(R,q,\beta) = Z\left(\frac{\alpha'}{R},\beta,q\right). \tag{2.26}$$

Note that the interchange of q and β corresponds to that of metric $G_{\varphi y}$ and B-field $B_{\varphi y}$, which is the essential part of T-duality transformation. Furthermore one can see the periodicity of q and β

$$Z(R,q,\beta) = Z\left(R, q+\frac{2n_1}{R}, \beta+\frac{2n_2 R}{\alpha'}\right), \qquad (n_1,n_2\in\mathbb{Z}). \tag{2.27}$$

2.3 Supersymmetry breaking and relation to type-0 theory

This Melvin model does not preserve supersymmetry except the case $(qR,\alpha'\beta/R)\in(2\mathbb{Z},2\mathbb{Z})$, which is equivalent to the ordinary type-II theory in the flat space by the periodicity (2.27). This fact can be seen from the non vanishing of the partition function $Z(R,q,\beta)$. This can also be understood at the level of supergravity. Let us assume[5] $\beta=0$ for simplicity, then all of spin 1/2 fermions which go around the circle $\mathbb{S}^1$ receive the phase factor[6] $e^{\pm i\pi qR}$. Thus only for $qR\in 2\mathbb{Z}$ the Killing spinors do exist and the supersymmetry is preserved. This shows the string theoretic realization of Scherk-Schwarz compactification [51, 52, 53]. In this kind of (spacetime) supersymmetry breaking the local supersymmetry is preserved while the global supersymmetry is broken [51]. This fact can be found in the open string spectrum on the D-brane as we will see in the section 3. For $\beta=0$ and $q\neq 0$ a D-brane which follows the Dirichlet boundary condition for $\mathbb{S}^1$ has the Bose-Fermi degeneracy while a D-brane which follows the Neumann boundary condition does not show the degeneracy.

Thus this model does not preserve supersymmetry in general and it may be unstable. Indeed it has tachyons if neither qR nor $\alpha'\beta/R$ is an integer [29]. In particular, for $(qR,\alpha'\beta/R)\in(2\mathbb{Z}+1,2\mathbb{Z})$, the model is identical to type-IIA(B) theory twisted by $(-1)^{F_S}\cdot\sigma_{1/2}$ with radius $2R$ as shown in [19], which is also equivalent to type-0B(A) theory twisted by $(-1)^{F_R}\cdot\sigma_{1/2}$ with radius α'/R [7]. Here the operators F_S, F_R and $\sigma_{1/2}$ represent the spacetime fermion number, the world-sheet right-moving fermion number and the half-shift operator on the circle $\mathbb{S}^1$. If we further take the small radius limit $R\to 0$, we obtain ten-dimensional type-0B(A) string theory [7] after we perform the T-duality. On the other hand, if we take the limit $R\to\infty$ with $\beta=0$, then the theory is identical to the ordinal ten-dimensional type-IIA(B) string theory.

[5] Here we restrict the range of β to $0\le\beta\alpha'/R<2$ because of its periodicity (2.27).

[6] This can be also seen if we construct the spin field from $\psi'_{L,R}$ with the boundary condition (2.21).

k	type-II orbifold (radius)	T-dualized orbifold (radius)
even	$\mathrm{IIA(B)}/\sigma_{1/N}\cdot g \quad (NR=\frac{N\alpha'}{\tilde{R}})$	$\mathrm{IIB(A)}/\tilde{\sigma}_{1/N}\cdot g \quad (\frac{\alpha'}{NR}=\frac{\tilde{R}}{N})$
odd	$\mathrm{IIA(B)}/\sigma_{1/2N}\cdot g \ (2NR=\frac{2N\alpha'}{\tilde{R}})$	$\mathrm{0B(A)}/\{(-1)^{F_R}\cdot\sigma_{1/2},\ \tilde{\sigma}_{1/N}\cdot g\}\ (\frac{\alpha'}{NR}=\frac{\tilde{R}}{N})$

Table 1: The equivalence of the IIA(B) Melvin background for $qR = k/N, \beta = 0$ with freely acting orbifolds. The operators $g = \exp(2\pi i \frac{k}{N}\hat{J})$, $\sigma_{1/N}$ and $\tilde{\sigma}_{1/N}$ represent the projection operator, $\frac{1}{N}$-shift operator in the circle and its dual operator, respectively.

This equivalence can be generalized to the Melvin background with the fractional values of the magnetic parameters [13]. For the specific fractional values[7] $qR = k/N, \beta = 0$ (or $\beta\alpha'/R = k/N, q = 0$) the Melvin model is equivalent to $\mathbb{Z}_N$ freely acting orbifolds. Furthermore if we take the limit $R \to 0$ (or $R \to \infty$), those are reduced to the abelian orbifolds $\mathbb{C}/\mathbb{Z}_N$ with a fixed point in type-II theory (for even k) and in type-0 theory (for odd k). We summarize the result in table 1.

3. D-branes in Melvin background

In this section we consider D-branes in Melvin background (2.2). In the previous section we have seen that the non-linear sigma model in the Melvin background can be exactly solved. By applying the T-duality in the curved space and by redefining appropriate target space variables, the non-linear sigma model can be rewritten by the free fields with the non-trivial boundary conditions, and we can quantize this action in the same way as that in the flat space.

With regard to open strings the quantization process can be also performed by the usual method. By changing the boundary conditions of open strings we can obtain the various D-branes[8] in the Melvin background, while the several constraints which can not be seen in the flat space arise due to the non triviality of the Melvin geometry. In the arguments below we define the Dp-brane as the D-brane which has $p+1$ Neumann boundary conditions and $9-p$ Dirichlet boundary conditions in terms of the free fields $(X', \bar{X}', Y')$, not the original fields $(X, \bar{X}, Y)$. The interpretation of such D-branes in the original coordinate $(X, \bar{X}, Y)$ of the Melvin background (2.2) is somewhat complicated as we will discuss later.

One of the interesting results in this section is that the D-brane spectra dramatically change according to whether the parameters in Melvin background qR and $\beta\alpha'/R$ take the rational or irrational value. For example if qR or $\beta\alpha'/R$ is rational there exist some D-branes which are movable in the $(X, \bar{X}) \in \mathbb{R}^2$ plane, while if both of these parameters take irrational values all D-branes are pinned at the origin in the Melvin geometry. The second interesting result is that some D-branes wrap the

[7]Here the integers k and N are assumed to be coprime.

[8]For $\beta = 0$ case the D-brane systems we discuss below are closely related to the D-branes in toroidal compactification of freely acting orbifolds [54].

non-trivial cycle in the Melvin geometry. For example, some D1-branes wrap the geodesic lines $\varphi + qy =$ constant in the Melvin geometry without NSNS B-field (2.3), which is the natural result because such a configuration gives the minimal mass of the D-brane.[9]

However, the most interesting argument about it is that these D1-branes have the "spiral" configuration. These wrap $\mathbb{S}^1$ in the Y direction, however the location of the end point of this D1-brane after wrapping one time on $\mathbb{S}^1$ is not the same as that of the start point. In other words its end point rotates from the starting point by some angle in the $(X, \bar{X}) \in \mathbb{R}^2$. After wrapping repeatedly, this configuration becomes something like a coil wrapping several times on the torus. If the parameter qR of the Melvin background is rational, the end point of its D1-brane after wrapping several times on S^1 connects to the starting point, while if the parameter is irrational its end point never meets the starting point and D1-brane wraps on the torus infinite times and becomes singular because the effective tension of D1-brane diverges.

There is the other motivation to consider D-branes in the Melvin background. As we have seen in the previous section or in [13], by tuning magnetic parameters qR and $\beta\alpha'/R$ appropriately the various ten-dimensional backgrounds which we already know can be realized, that is, type II, type 0 in flat space and these abelian orbifolds $\mathbb{C}/\mathbb{Z}_N$. Therefore, if we consider D-branes in the Melvin background, we may understand how the D-branes in the above various backgrounds are connected with each other.

In orbifold theories there exist two types of D-branes which are called the fractional D-brane and the bulk D-brane [32, 33]. What we will find is that the fractional D-branes and bulk D-branes correspond to the pinned and movable D-branes which we have mentioned, respectively. Moreover these D-branes prove to naturally correspond not only to the D-branes in type-II theory in flat space but also the bound state of an electric D-brane and a magnetic D-brane in type-0 theory [34]–[37] and [55, 56].[10] We can verify this by the calculation of the tension and the vacuum amplitude etc. by the method of the boundary state formalism.

3.1 Boundary state in Melvin background

In order to investigate D-branes in general conformal field theories, it is convenient to use the boundary state formalism. The boundary state is one way of the representations of D-branes in the closed string Hilbert space, and thus from this we can investigate the interactions between two D-branes and between a D-brane and closed strings.

From previous sections we have already known that the action for the Melvin background reduces to that written by free fields. Therefore the construction of the boundary state is similar to that in the flat space (see [57, 58] and references there in), or more precisely to that in orbifold theories [35, 59, 60, 61] even though the

[9]In the Melvin background with B-field the analysis becomes a little complicated, and we do not verified it in this paper.

[10]In type-0 theory two kinds of D-branes do appear as we will review later.

D-branes which we will construct have various new intriguing structures. Below we will omit the trivial oscillators for $\mathbf{R}^{1,6}$ directions on the world-sheet. One can use either the light-cone or covariant formulation.

For the compactified direction Y', the usual Neumann or Dirichlet boundary conditions are allowed:

$$\text{Neumann}: \begin{cases} \partial_\tau Y'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \left(\frac{n}{R} - q\hat{J}\right)|B\rangle = 0\,, \\ \qquad\qquad\qquad\qquad\qquad \left(\beta_m + \tilde{\beta}_{-m}\right)|B\rangle = 0\,, \\ (\eta'_L - \epsilon\eta'_R)|_{\tau=0}|B\rangle = 0 \longleftrightarrow (\eta_r - i\epsilon\tilde{\eta}_{-r})|B\rangle = 0\,, \end{cases} \tag{3.1}$$

$$\text{Dirichlet}: \begin{cases} \partial_\sigma Y'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \left(\frac{Rw}{\alpha'} - \beta\hat{J}\right)|B\rangle = 0\,, \\ \qquad\qquad\qquad\qquad\qquad \left(\beta_m - \tilde{\beta}_{-m}\right)|B\rangle = 0\,, \\ (\eta'_L + \epsilon\eta'_R)|_{\tau=0}|B\rangle = 0 \longleftrightarrow (\eta_r + i\epsilon\tilde{\eta}_{-r})|B\rangle = 0\,, \end{cases} \tag{3.2}$$

where $|B\rangle$ is the boundary state and the parameter ϵ takes the value ± 1 which comes from open string boundary conditions for world-sheet fermions. Note that the zero mode for Y' is given by (2.20). For the directions $X', \bar{X}'$ the allowed boundary conditions are

$$\text{Neumann-Neumann}: \begin{cases} \partial_\tau X'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad (\alpha_{m-\gamma} + \tilde{\alpha}_{-m+\gamma})|B\rangle = 0\,, \\ \partial_\tau \bar{X}'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad (\bar{\alpha}_{m+\gamma} + \tilde{\bar{\alpha}}_{-m-\gamma})|B\rangle = 0\,, \\ (\psi'_L - \epsilon\psi'_R)|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad \left(\psi_{r-\gamma} - i\epsilon\tilde{\psi}_{-r+\gamma}\right)|B\rangle = 0\,, \\ \left(\bar{\psi}'_L - \epsilon\bar{\psi}'_R\right)|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad \left(\bar{\psi}_{r+\gamma} - i\epsilon\tilde{\bar{\psi}}_{-r-\gamma}\right)|B\rangle = 0\,, \end{cases} \tag{3.3}$$

$$\text{Dirichlet-Dirichlet}: \begin{cases} \partial_\sigma X'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad (\alpha_{m-\gamma} - \tilde{\alpha}_{-m+\gamma})|B\rangle = 0\,, \\ \partial_\sigma \bar{X}'|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad (\bar{\alpha}_{m+\gamma} - \tilde{\bar{\alpha}}_{-m-\gamma})|B\rangle = 0\,, \\ (\psi'_L + \epsilon\psi'_R)|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad \left(\psi_{r-\gamma} + i\epsilon\tilde{\psi}_{-r+\gamma}\right)|B\rangle = 0\,, \\ \left(\bar{\psi}'_L + \epsilon\bar{\psi}'_R\right)|_{\tau=0}|B\rangle = 0 \longleftrightarrow \\ \qquad\qquad \left(\bar{\psi}_{r+\gamma} + i\epsilon\tilde{\bar{\psi}}_{-r-\gamma}\right)|B\rangle = 0\,, \end{cases} \tag{3.4}$$

where γ is given by (2.22).[11] Here we have to note that the Neumann-Dirichlet or Dirichlet-Neumann boundary conditions can be defined only if γ takes integer or

[11]At this stage we can take the more general boundary conditions for complex fermions (ϵ can take U(1) complex values and can be unequal to ϵ in (3.1) and (3.2)). However, if we consider the $\mathcal{N} = 1$ superconformal invariance (3.5), such boundary conditions are not allowed.

half-integer values. We will discuss these special cases in the last of next subsection. We would also like to stress that in the above arguments we have defined the Dirichlet or Neumann boundary conditions with respect to the free fields $(X', \bar{X}', Y')$. These boundary conditions are not always equivalent to the Dirichlet or Neumann boundary conditions with respect to the original fields $(X, \bar{X}, Y)$ in the Melvin sigma model (2.4) as we will see later.

From these conditions and from (B.3) and (B.4) we can verify that the boundary state defined by (3.1) $\sim$ (3.4) satisfies the $\mathcal{N}=1$ superconformal invariance

$$\left(L_m - \tilde{L}_{-m}\right)|B\rangle = 0\,, \qquad \left(G_r + i\epsilon\tilde{G}_{-r}\right)|B\rangle = 0\,. \tag{3.5}$$

Moreover from (3.3) and (3.4) we can verify

$$\hat{J}|B\rangle = \left(\hat{J}_L + \hat{J}_R\right)|B\rangle = 0\,, \tag{3.6}$$

where the mode expansions of $\hat{J}_L$ and $\hat{J}_R$ are given by (B.6). This shows that these D-branes (NN or DD boundary condition) preserve the rotational symmetry on the plane $\mathbb{R}^2$ as expected.[12]

Now we can write down the boundary state explicitly. From (3.1) $\sim$ (3.4) its explicit form is almost the same as that in the flat space [57, 58], except the shift of oscillator indices by $\hat{\gamma}$ and the GSO-projection. For example, the NSNS sector of the boundary state for a D0-brane at $\rho = 0$ is given by

$$\begin{aligned}
|B,\gamma,\epsilon\rangle_{\mathrm{NSNS}} = \exp&\left[\sum_{m=1}^{\infty} m^{-1}\beta_{-m}\tilde{\beta}_{-m}\right]|n,0\rangle \otimes \exp\left[-i\epsilon\sum_{r=1/2}^{\infty}\eta_{-r}\tilde{\eta}_{-r}\right]|0\rangle \otimes \\
\otimes \exp&\left[\sum_{m=0}^{\infty}(m+\hat{\gamma})^{-1}\alpha_{-m-\hat{\gamma}}\tilde{\bar{\alpha}}_{-m-\hat{\gamma}} + \sum_{m=1}^{\infty}(m-\hat{\gamma})^{-1}\bar{\alpha}_{-m+\hat{\gamma}}\tilde{\alpha}_{-m+\hat{\gamma}}\right]|0\rangle_{\hat{\gamma}} \otimes \\
\otimes \exp&\left[-i\epsilon\sum_{r=1/2}^{\infty}\left\{\psi_{-r-\hat{\gamma}}\tilde{\bar{\psi}}_{-r-\hat{\gamma}} + \bar{\psi}_{-r+\hat{\gamma}}\tilde{\psi}_{-r+\hat{\gamma}}\right\}\right]|0\rangle_{\hat{\gamma}+\frac{1}{2}}\,,
\end{aligned} \tag{3.7}$$

where γ is equal to $\beta\alpha' n/R$ $(n \in \mathbb{Z})$ as can be seen from eqs. (2.22) and (3.2), and $\hat{\gamma}$ is defined in eq. (2.24).[13] In the above expression we have abbreviated the trivial

[12] More precisely, one should take into account the bosonic zero-mode contribution to the angular momentum $\hat{J}_0 = i\sqrt{2/\alpha'}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ if γ takes an integer value. However, we can neglect this if the D-brane obeys the Neumann-Neumann (NN) boundary condition for $(X', \bar{X}')$ or if the D-brane obeys the Dirichlet-Dirichlet (DD) boundary condition located at $\rho = 0$. Even if we consider a D-brane with the DD boundary condition and move it away from the origin $\rho = 0$, we can choose n or w such that $(\frac{n}{R} - q\hat{J}_0)|B\rangle = 0$ or $(\frac{Rw}{\alpha'} - \beta\hat{J}_0)|B\rangle = 0$, respectively. We will return this point in section 3.2.

[13] We have assumed the specific range $0 < \hat{\gamma} < 1/2$. The extension of this expression to the other range of $\hat{\gamma}$ and to RR-sector is straightforward.

part which comes from the other directions than $X', \bar{X}'$ and Y'. Boundary states for D-branes which obey other boundary conditions can be obtained similarly. The total boundary state is given by $|B\rangle = |B\rangle_{\rm NSNS} \pm |B\rangle_{\rm RR}$. The plus (minus) sign corresponds to a D-brane (an anti D-brane).

Next we have to consider the closed string GSO-projection. This is somewhat non trivial because as we said in the lines below (2.25) the GSO-projection for $2n \leq \gamma < 2n+1$ ($n \in \mathbb{Z}$) is the usual projection for type-II theory, while for $2n+1 \leq \gamma < 2n+2$ ($n \in \mathbb{Z}$) it is the projection *with the additional minus sign*. Thus, the GSO-invariant boundary state is represented by

$$\begin{aligned}
|B\rangle_{\rm NSNS} &= \frac{1}{2\pi R}\sum_{n=-\infty}^{\infty} f_\gamma \ \frac{1+(-1)^{[\gamma]}(-1)^{F_L}}{2}\frac{1+(-1)^{[\gamma]}(-1)^{F_R}}{2}|B,\gamma,+\rangle_{\rm NSNS} \\
&= \frac{1}{2(2\pi R)}\sum_{n=-\infty}^{\infty} f_\gamma \left[|B,\gamma,+\rangle_{\rm NSNS} - (-1)^{[\gamma]}|B,\gamma,-\rangle_{\rm NSNS}\right], \\
|B\rangle_{\rm RR} &= \frac{1}{2\pi R}\sum_{n=-\infty}^{\infty} f_\gamma \ \frac{1+(-1)^{[\gamma]}(-1)^{F_L}}{2}\frac{1\mp(-1)^{[\gamma]}(-1)^{F_R}}{2}|B,\gamma,+\rangle_{\rm RR} \\
&= \frac{1}{2(2\pi R)}\sum_{n=-\infty}^{\infty} f_\gamma \left[|B,\gamma,+\rangle_{\rm RR} + (-1)^{[\gamma]}|B,\gamma,-\rangle_{\rm RR}\right], \\
&\qquad\qquad \left(\mp = \begin{cases} -\text{for IIA}\,, \\ +\text{for IIB}\,, \end{cases}\right)
\end{aligned} \tag{3.8}$$

where $[\gamma]$ is the Gauss symbol which picks up the maximal integer part of γ, and constants f_γ in (3.8) are determined by the Cardy's condition [62]. This is the consistency condition that the vacuum amplitude between two D-branes computed in the closed string sector by using the boundary state should be equal to the cylinder amplitude of open string.

Therefore we would like to calculate the vacuum amplitude by using the above boundary state. For a D0-brane we obtain the following result[14][15]

$$\begin{aligned}
A = \frac{\alpha'}{8\pi R}V_0 \sum_{\gamma\in\mathbb{Z}} \ |f_\gamma|^2 \ \int_0^\infty ds \ (2\pi\alpha' s)^{-4} \exp\left(-\frac{s\alpha' n^2}{2R^2}\right)(\eta(\tau))^{-12} \times \\
\times \left[(\theta_3(0|\tau))^4 - (-1)^\gamma(\theta_4(0|\tau))^4 - (\theta_2(0|\tau))^4\right] +
\end{aligned}$$

[14] Here we have used the explicit form of L_0 and $\tilde{L}_0$ in (B.4) for the closed string propagator. We also have employed the quasi periodicity of theta functions (C.3).

[15] Note that we have divided γ into the integer part and the other part, because if γ takes an integer value the naive calculation of the vacuum amplitude by using (3.7) diverges. This can be seen in the third line of eq. (3.9) if we notice the relation $\theta_1(\nu|\tau) = 0$ ($\gamma \in \mathbb{Z}$) by using (C.3). This divergence is due to the reappearance of the zero modes of $X', \bar{X}'$, which can be seen from (2.21) and (B.1). Therefore, the expression of the boundary state such as (3.7) is not correct for integer γ, and we have to redefine its bosonic part for $X', \bar{X}'$.

$$+\frac{i\alpha'}{8\pi R}V_0\sum_{\gamma\notin\mathbb{Z}}(-1)^{[\gamma]}\,|f_\gamma|^2\int_0^\infty ds\,\left(2\pi\alpha' s\right)^{-3}\times$$

$$\times\,\exp\left(-\frac{s\alpha' n^2}{2R^2}\right)\left(\eta(\tau)\right)^{-9}\left(\theta_1(\nu|\tau)\right)^{-1}\times$$

$$\times\left[(\theta_3(0|\tau))^3(\theta_3(\nu|\tau))-(\theta_4(0|\tau))^3(\theta_4(\nu|\tau))-(\theta_2(0|\tau))^3(\theta_2(\nu|\tau))\right],\qquad(3.9)$$

where $\tau = is/\pi, \nu = i\gamma s/\pi$. The volume factor for the time-direction is denoted by V_0. By replacing s with π/t and using the modular transformations for theta functions eq. (C.2) we can obtain the following result

$$A=\frac{\pi\alpha'}{(8\pi R)(2\pi^2\alpha')^4}V_0\sum_{\gamma\in\mathbb{Z}}|f_\gamma|^2\int_0^\infty\frac{dt}{t^2}\,\exp\left(-\frac{\pi\alpha' n^2}{2R^2t}\right)(\eta(it))^{-12}\times$$

$$\times\,\left[(\theta_3(0|it))^4-(-1)^\gamma(\theta_2(0|it))^4-(\theta_4(0|it))^4\right]+$$

$$+\frac{\pi\alpha'}{(8\pi R)(2\pi^2\alpha')^3}V_0\sum_{\gamma\notin\mathbb{Z}}(-1)^{[\gamma]}\,|f_\gamma|^2\times$$

$$\times\int_0^\infty\frac{dt}{t^2}\,\exp\left(-\frac{\pi\alpha' n^2}{2R^2t}\right)(\eta(it))^{-9}(\theta_1(\gamma|it))^{-1}\times$$

$$\times\,\left[(\theta_3(0|it))^3(\theta_3(\gamma|it))-(\theta_2(0|it))^3(\theta_2(\gamma|it))-(\theta_4(0|it))^3(\theta_4(\gamma|it))\right].\qquad(3.10)$$

On the other hand the vacuum amplitude can be obtained also from the open string one-loop calculation

$$Z_O=\int_0^\infty\frac{dt}{2t}\,\mathrm{Tr}_{\mathrm{NS-R}}\left[\frac{1+(-1)^F}{2}q^{H_O}\right],\qquad\left(q\equiv e^{-2\pi t}\right),\qquad(3.11)$$

where we have defined[16] $\mathrm{Tr}_{\mathrm{NS-R}}=\mathrm{Tr}_{\mathrm{NS}}-\mathrm{Tr}_{\mathrm{R}}$; the operator H_O denotes the open string hamiltonian[17]

$$H_O=\alpha' p^2+\alpha'\left(\frac{Rw}{\alpha'}-\beta\hat{J}\right)^2+\hat{N}\,,\qquad(3.12)$$

where $\hat{N}$ and $\hat{J}$ represent the occupation number operator including the zero point energy ($-1/2$ for NS-sector and 0 for R-sector) and the angular momentum generator in $\mathbb{R}^2$ plane both of which are the open string analog of (B.5) and (B.6), respectively. However, this open string system has two crucially different points from closed string one. The first is that there are no tachyons on D-branes as we can see from the above equations, while there are in the closed string spectrum (2.23). The second is that the indices of modes for $X', \bar{X}', Y'$ and their superpartners take *the integer or half-integer* (for NS-sector) values because the boundary conditions of open strings obey

[16]The trace includes the factor two due to the Chan-Paton factor.

[17]We are now in the T-dualized space where D0-branes live, thus only the winding mode w, not the momentum mode n, appears in this open string hamiltonian.

usual Neumann or Dirichlet conditions, not twisted ones (2.21). In particular the mode expansion of $\hat{J}$ by open string NS-modes is

$$\hat{J} = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0) + \sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}\bar{\alpha}_n - \bar{\alpha}_{-n}\alpha_n) + \sum_{r=1/2}^{\infty}\left(\psi_{-r}\bar{\psi}_r - \bar{\psi}_{-r}\psi_r\right), \quad (3.13)$$

where x_0 and $\bar{x}_0$ are the zero modes for X' and $\bar{X}'$, respectively.[18] For R-sector the indices of ψ and $\bar{\psi}$ run integer values, and its zero mode contribution $\frac{1}{2}[\psi_0, \bar{\psi}_0]$ should be added. Therefore we can see that the eigenvalues of $\hat{J}$ take integer values for NS-sector and half-integer values for R-sector being consistent with the spin-statistics relation.

Now let us apply the Cardy's condition [62] (or open-closed duality). By using the Poisson resummation formula the open string amplitude (3.11) becomes

$$\begin{aligned} A = \frac{2\pi\alpha'}{R}\int_0^{\infty}\frac{dt}{2t}(8\pi^2\alpha' t)^{-1/2}\,\mathrm{Tr}_{\mathrm{NS-R}}\Bigg[\frac{1+(-1)^F}{2}\, q^{\alpha' p^2 + \hat{N}} \times \\ \times \sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi\alpha'}{2R^2 t}n^2 + 2\pi i\frac{\beta\alpha'}{R}n\hat{J}\right)\Bigg]. \end{aligned} \quad (3.14)$$

By requiring the equality between (3.10) and (3.14) we obtain

$$f_\gamma = \begin{cases} \frac{1}{2}T_0, & (\gamma \in \mathbb{Z}), \\ \frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma|}{2\pi^2\alpha'}\right)^{1/2} T_0\,, & (\gamma \notin \mathbb{Z}), \end{cases} \quad (3.15)$$

where we have defined $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$ and T_p/κ (κ is the gravitational coupling constant) is equal to the tension of an ordinary type-II Dp-brane in flat space. For more general Dp-branes the computations can be performed in the same way. Its open string hamiltonian is the same as eq. (3.12) except the reappearance of the zero modes.[19] The value of f_γ is given by

$$f_\gamma = \begin{cases} \frac{1}{2}T_p\,, & (\gamma \in \mathbb{Z}), \\ \frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma|}{2\pi^2\alpha'}\right)^{\mp 1/2} T_p\,, & (\gamma \notin \mathbb{Z}). \end{cases} \quad (3.16)$$

[18] Especially for D0-branes which we are considering here the orbital angular momentum part $\hat{J}_0 = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ should be neglected due to the absence of the open string zero mode.

[19] The non-trivial relation is the trace formula which comes from the open string zero modes of X' and $\bar{X}'$. This is used for D-branes which obey the Neumann-Neumann boundary condition for $X', \bar{X}'$ directions:

$$\mathrm{Tr}\ \exp\left[-2\pi\alpha' tp^2 + 2\pi i\gamma\hat{J}_0\right] = \begin{cases} (2\sin\pi\gamma)^{-2}\,, & (\gamma \notin \mathbb{Z}), \\ V_2\ (8\pi^2\alpha' t)^{-1}\,, & (\gamma \in \mathbb{Z}), \end{cases}$$

where $\hat{J}_0 = \frac{i}{\sqrt{2\alpha'}}(x_0\bar{\alpha}_0 - \bar{x}_0\alpha_0)$ is the orbital angular momentum, and V_2 is the volume factor for $(X', \bar{X}') \in \mathbb{R}^2$ plane. Such a trace is familiar in orbifold theories (see for example [63]).

The sign factors $\mp$ take $-$ for Dp-branes with the Neumann-Neumann boundary condition for $X, \bar{X}'$ directions, $+$ for Dp-branes with the Dirichlet-Dirichlet boundary condition for those directions.

The above results for $\gamma \in \mathbb{Z}$ show that a Dp-brane in the Melvin background has the same tension as that in the flat space. We would also like to note that no open string tachyonic modes appear on these Dp-branes.

3.2 Structure of D-branes for the rational parameters

As we have said in section 2, the nature of the Melvin background depends sensitively on whether the (dimensionless) magnetic parameters qR and $\beta\alpha'/R$ take the rational or irrational values. Especially in the former case with $qR = k/N$, $\beta = 0$ (or $\beta\alpha'/R = k/N$, $q = 0$) this background is equivalent to the freely acting orbifold,[20] and under the limit $R \to 0$ ($R \to \infty$) it is reduced to the abelian orbifold $\mathbb{C}/\mathbb{Z}_N$ in type II (for even k) or in type 0 (for odd k) [13].

In this section we consider D-branes in the Melvin background with the rational parameter. As we will see below even for the finite radius a single D-brane and N different kinds of D-branes have similar properties to a fractional D-brane and a bulk D-brane in orbifold theories, respectively.[21] Therefore, from now on, we will often call these two types of D-branes in the Melvin background the fractional D-brane and the bulk D-brane.

Fractional D0-branes. Let us first discuss a D0-brane, which has the Dirichlet boundary condition in both $\mathbb{R}^2$ and $\mathbb{S}^1$ direction. As can be seen from its boundary state (3.7), its behavior depends only on $\gamma = \frac{\beta\alpha'}{R}n$. Moreover we can find in (3.7) that there are no zero-modes in $X', \bar{X}'$ directions unless γ takes an integer value. We can equally say that this D0-brane can not leave from $X' = \bar{X}' = 0$ unless $\gamma \in \mathbb{Z}$, and thus the D0-brane is expected to become a fractional D0-brane [32, 33, 59] on the orbifolds $\mathbb{C}/\mathbb{Z}_N$ in the limit $R \to \infty$, $\beta\alpha'/R = k/N$ and $qR \to 0$. To verify this we begin with the analysis of this orbifold limit.

First we take this limit for the boundary state (3.8). Here we reparametrize the momentum number as $n = N\alpha + l$ $\;(\alpha \in \mathbb{Z},\ l = 0, 1, \ldots, N-1)$. In this limit the NSNS-sector of the boundary state (3.8) becomes

$$\lim_{R\to\infty} \frac{1}{2(2\pi R)} \sum_{l=0}^{N-1} \sum_{\alpha\in\mathbb{Z}} e^{i\frac{y'}{R}(N\alpha+l)} f_\gamma \times \\ \times \left[\left| B, \frac{kl}{N}, + \right\rangle_{\text{NSNS}} - (-1)^{k\alpha+[kl/N]} \left| B, \frac{kl}{N}, - \right\rangle_{\text{NSNS}} \right] =$$

[20] The discussion includes the $N = 1$ case which is equivalent to the type-0 theory with $\mathbb{Z}_2$ twist $(-1)^{F_R} \cdot \sigma$ [7]. For earlier discussions on the D-branes in this model see [56, 64].

[21] Note that our open string results hold if either qR or $\beta\alpha'/R$ is rational and another is arbitrary. In general the closed string theory with these values can not be identified with the freely acting orbifold [13].

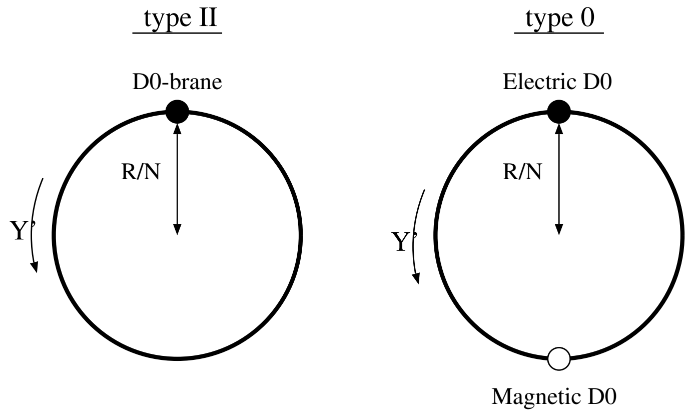

Figure 1: D0-branes

$$
= \begin{cases} \frac{\delta(y')}{2} \sum_{l=0}^{N-1} f_l' \left[\left|B, \frac{kl}{N}, +\right\rangle_{\mathrm{NSNS}} - (-1)^{[kl/N]} \left|B, \frac{kl}{N}, -\right\rangle_{\mathrm{NSNS}} \right], & (\text{even k}), \\ \frac{\delta(y')}{2} \sum_{l=0}^{N-1} f_l' \ \left|B, \frac{kl}{N}, +\right\rangle_{\mathrm{NSNS}} - \frac{\delta\left(y' - \frac{\pi R}{N}\right)}{2} \times & \\ \quad \times \sum_{l=0}^{N-1} (-1)^{[kl/N]} f_l' \ \left|B, \frac{kl}{N}, -\right\rangle_{\mathrm{NSNS}}, & (\text{odd k}), \end{cases} \tag{3.17}
$$

$$
\text{where} \qquad f_l' = \begin{cases} \frac{1}{2N} T_0 \,, & (l = 0)\,, \\ \frac{1}{\sqrt{2}N} \left(\frac{|\sin \pi k l/N|}{2\pi^2 \alpha'} \right)^{1/2} T_0 \,, & (l \neq 0)\,, \end{cases} \tag{3.18}
$$

and the RR-sector of the boundary state can be obtained in the same way. Note that here we extract the α dependent factor from the boundary state (3.7) which comes only from the zero mode contribution of Y' as $|n, 0\rangle = \exp\left(i\frac{n}{R}y'\right)|0\rangle$.

For even k this is just the boundary state for a fractional D0-brane in the type-II orbifold $\mathbb{C}/\mathbb{Z}_N$.[22] Indeed we can identify the summation over l as the contribution from one untwisted sector and $N-1$ twisted sectors. Moreover the coefficient f_l' in eq. (3.18) is $1/N$ times of f_γ in eq. (3.15) and this shows the fractional nature of this D-brane explicitly.

For odd k the boundary state is a little more complicated than for even k. The most important argument is that two kinds of D-branes appear each at $y' = 0$ (the first term in (3.17)) and at $y' = \pi R/N \to \infty$ (the second term in (3.17)). The former corresponds to an electric fractional D0-brane and the latter a magnetic fractional D0-brane in the orbifold $\mathbb{C}/\mathbb{Z}_N$ of type-0 theory (see figure 1). The fractional nature of the D-brane can be understood in the same way as that in type-II theory, and the appearance of two kinds of D-branes is well known in type-0 theory [7, 34, 35, 36,

[22]Note that in (3.17) we have not used the necessary condition $qR \to 0$ to realize the orbifolds $\mathbb{C}/\mathbb{Z}_N$ since the boundary state (3.17) does not depend on the value of q. This is different from the closed string theory which depends on both of the parameters q, β. We will discuss the meaning of this in section 3.4.

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55, 56].[23] To explain it let us take the flat type-0 limit $N = 1$ (see section 2.3) in the boundary state (3.17). Then the untwisted sector ($l = 0$) only remains, and if we divide it into one with $\delta(y')$ and another with $\delta(y' - \pi R)$ those boundary states are given by as follows

$$\begin{aligned} |\text{electric}\rangle &= \frac{T_0}{4}\left(|B,0,+\rangle_{\rm NSNS} + |B,0,+\rangle_{\rm RR}\right), \\ |\text{magnetic}\rangle &= \frac{T_0}{4}\left(-|B,0,-\rangle_{\rm NSNS} + |B,0,-\rangle_{\rm RR}\right). \end{aligned} \tag{3.19}$$

These are just the boundary states for an electric D0-brane and a magnetic D0-brane in the type-0 theory.[24] Note that both of the boundary states (3.19) are invariant under the diagonal GSO-projection $(1 + (-1)^{F_L+F_R})/2$ which appears in the closed string theory of type 0. The appearance of two kinds of D-branes can be understood from the fact that the number of massless RR-fields in type-0A(B) theory is two times as many as those in type IIA(B).[25]

The identification with fractional D-branes can also be shown explicitly by examining the vacuum amplitude (3.14). By taking the orbifold limit, the summation part of NS-sector in (3.14) becomes

$$\begin{aligned} &\lim_{R\longrightarrow\infty}\sum_{l=0}^{N-1}\sum_{\alpha\in\mathbb{Z}}\exp\left[-\frac{\pi\alpha'}{2R^2t}(N\alpha+l)^2 + 2\pi i\frac{k}{N}(N\alpha+l)\hat{J}\right] = \\ &= R\int_{-\infty}^{\infty}dp\ \exp\left(-\frac{\pi\alpha' N^2}{2t}p^2\right)\sum_{l=0}^{N-1}\exp\left[2\pi i\frac{k}{N}l\hat{J}\right] \\ &= \frac{R}{2\pi\alpha'}\left(8\pi^2\alpha' t\right)^{1/2}\frac{\sum_{l=0}^{N-1}\ g^l}{N}. \end{aligned} \tag{3.20}$$

where $g = \exp[2\pi i\frac{k}{N}\hat{J}]$. Here we have used the fact that for NS-sector the eigenvalues of $\hat{J}$ take integers. On the other hand for R-sector $\hat{J}$ takes the half-integers, and the

[23] Of course the two kinds of the corresponding anti D-branes also exist even though in this paper we do not discuss those in the Melvin background.

[24] Strictly speaking the coefficient of the boundary state is a little different. The correct coefficient of the boundary state for a type-0 D-brane is $\sqrt{2}$ times as large as one in (3.19). This mismatch comes from the fact that the amplitude (3.21) counts the open string degrees of freedom *on one D-brane*, while in (3.17) *two D-branes* appear the infinite distance away from each other, whose configuration seems singular in the uncompactified theory. Thus if we consider *one D-brane* in the uncompactified theory (an electric or a magnetic D-brane) we have to multiply $\sqrt{2}$ in front of the boundary state. Such a multiplication matches with the fact that the tension of a type-0 D-brane is $1/\sqrt{2}$ times as large as that of a type-II D-brane [34].

[25] Remember that the definition of type-0 theory is the superstring theory with the unusual diagonal GSO-projection (see [6]), and its spectrum is given by

$$\begin{aligned} 0A:&\quad (NS+,NS+)\ (NS-,NS-)\ (R+,R-)\ (R-,R+)\,, \\ 0B:&\quad (NS+,NS+)\ (NS-,NS-)\ (R+,R+)\ (R-,R-)\,, \end{aligned}$$

where the sign $\pm$ is denoted by the world-sheet spinor number $(-1)^{F_L}$ and $(-1)^{F_R}$.

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phase factor $\exp(2\pi i k\alpha\hat{J}) = (-1)^{k\alpha}$ appears in the summation $\sum_{\alpha\in\mathbb{Z}}$. From this we can see that the amplitude in the R-sector becomes zero for odd k. Therefore the vacuum amplitude (3.14) under the orbifold limit becomes

$$\begin{aligned}
&\int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}_{\mathrm{NS-R}}\left[\frac{1+(-1)^F}{2}\,\frac{\sum_{l=0}^{N-1}\, g^l}{N}\; q^{H'_O}\right], \qquad (\text{for even } k)\,,\\
&\int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}_{\mathrm{NS}}\left[\frac{1+(-1)^F}{2}\,\frac{\sum_{l=0}^{N-1}\, g^l}{N}\; q^{H'_O}\right], \qquad (\text{for odd } k)\,,
\end{aligned} \tag{3.21}$$

where $H'_O = \alpha' p^2 + \hat{N}$ is the open string hamiltonian. This is just the open string amplitude of a fractional D0-brane in type II for even k and that in type 0 for odd k (for fractional D-branes in type-0 theory see also [35]). In particular the absence of R-sector for the type 0 can be understood from the fact that there are no fermions on a type-0 D-brane, which can be verified by using the boundary state (3.19). Although the R-sector in type 0 can appear from the open strings between an electric D-brane and a magnetic D-brane, these two D-branes in (3.17) are infinitely far away from each other and its spectrum is neglected.

Let us return to the case of the finite radius R. From the above arguments the D-branes are expected to be similar to the fractional D0-branes in the orbifold theories. Indeed it is easy to see that the corresponding D0-branes are stuck at the point $(X'{=}\bar{X}'{=}0)$ since there are no zero-modes of $X', \bar{X}'$ for non integer $\gamma = \frac{\beta\alpha'}{R}n = \frac{k}{N}n$ (see (2.21) and (B.1)).

Up to now we have not considered the U(1) phase in the coefficient f_γ (3.15) of the boundary state (3.8). We can consider the freedom of the translation of D0-branes in the compactified Y' direction and this effect can be included in the boundary state (3.8) by replacing f_γ with $f_\gamma \exp(i\frac{n}{R}y'_0)$ $(0 \le y'_0 < 2\pi R)$.[26] To consider its meaning in the orbifold picture, where it is appropriate to regard the radius of $\mathbb{S}^1$ as R/N (see table 1 and eq. (3.17)), we reparametrize y'_0 as $y'_0 = \tilde{y}'_0 + \frac{2\pi R}{N}a$ $(0 \le \tilde{y}'_0 < 2\pi R/N,\ a = 0, 1, \ldots, N-1)$. Then its boundary state is written by

$$|B_a, \epsilon\rangle_{\mathrm{NSNS,RR}} = \sum_{n\in\mathbb{Z}} e^{i\frac{n}{R}\tilde{y}'_0 + 2\pi i \frac{a}{N}n}\,|B, \gamma, \epsilon\rangle_{\mathrm{NSNS,RR}}\,, \qquad (\epsilon = \pm 1)\,. \tag{3.22}$$

If we take the orbifold limit $R \to \infty$, eq. (3.22) becomes eq. (3.17) except that $\delta(y')$ and f'_l are replaced by $\delta(y' - \tilde{y}'_0)$ and $f'_l \exp(2\pi i \frac{a}{N} l)$. Then if we remember that the fractional D-branes in an orbifold theory are labeled by irreducible representations of its discrete group [32, 33], we can see that these boundary states $(a = 0, 1, \ldots, N-1)$ represent the N types of fractional D0-branes in the orbifold $\mathbb{C}/\mathbb{Z}_N$. Moreover, we can see that the translation of the D0-brane by $2\pi R/N$ in the Y' direction is equivalent to changing the types of the fractional D0-brane in the orbifold picture since this

[26] This can be regarded as the Wilson line on a D1-brane in the T-dualized picture.

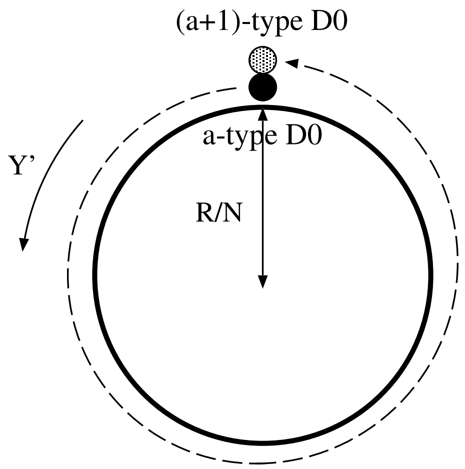

Figure 2: A fractional D0-brane receives the monodromy which changes its type.

manipulation is equivalent to $a \to a + 1$ in (3.22). This can be equally said that in the orbifold picture with the radius R/N the N types of fractional D0-branes are put at the same point $Y' = \tilde{y}'_0$, and receive the monodromy to change their types into each other if they go around $\mathbb{S}^1$ (see figure 2).

This observation may be also supported by the mass spectrum of open strings between a a-type and a b-type D0-brane (we set $\tilde{y}'_0 = 0$ for simplicity)

$$\alpha' M^2 = \frac{R^2}{N^2 \alpha'} \left(Nw - \left(k\hat{J} - a + b \right) \right)^2 + \hat{N} \,. \tag{3.23}$$

For even k the energy due to winding modes cannot vanish unless $a - b \in \mathbb{Z}$ and this shows that there are only N types of D0-branes coincident at an arbitrary point in the orbifold picture. It would be useful if we could clarify the geometrical picture of the D0-brane in terms of the original coordinate Y, not the free field Y', of Melvin background (2.4). This is complicated due to the T-duality transformations in section 2.1. We can speculate that the D0-brane, which is localized in the Y' direction, seems to be extended in the direction Y. This may be regarded as the $\rho = 0$ limit of a torus-like D2-brane (see figure 3) which we will discuss later.

For odd k the energy for NS-sector due to winding modes in (3.23) can vanish for the appropriate values of w, a, b and $\hat{J}$ (for example $w = a = b = \hat{J} = 0$), while for R-sector it can not because $\hat{J}$ takes the half-integer values and there remains non-zero minimal energy $(R/2N\alpha')^2$. This shows that a something like a bound system of an electric and a magnetic D0-brane exists in the Melvin background, being the finite distance $\pi R/N$ away from each other as we have already speculated from eq. (3.17).[27] These interpretations of D0-branes are shown in figure 1.

[27] Of course these are not exactly the same as an electric and a magnetic D0-brane in type-0 theory, and probably one dissolves into another because we can not split the boundary state (3.8)

Finally we would like to mention the another orbifold limit $R \to 0$ with $qR = k/N$ and $\beta = 0$. In this case we again obtain the orbifold $\mathbb{C}/\mathbb{Z}_N$. However, the boundary state (3.7) represents a bulk D0-brane in this orbifold limit since it depends only on β. Thus one may ask whether there exists a D0-brane[28] for finite R which is reduced to a fractional D0-brane in the limit $R \to 0$. Even though this is an intriguing problem, it will be beyond the scope of the present paper.

Bulk D0-branes. The fractional D0-branes are the most fundamental D0-branes in orbifold theories, and other D0-branes can be constructed by the linear combinations of them. In general these D0-branes can not leave from the fixed point $\rho = 0$, while if we collect N different types of fractional D0-branes they can move as an unit such that $\rho \neq 0$. The latter can be regarded as another type of the D0-brane in the orbifold theories which is called a bulk D0-brane, and it is known that the Chan-Paton bundle on this D0-brane obeys the regular representation for the discrete group $\mathbb{Z}_N$ [32].

Then it is natural to ask if such a D0-brane exists for the finite R. The answer is yes if the parameter $\beta\alpha'/R$ is rational, and the bulk D0-brane in the Melvin background is defined as a bound state of N different fractional D-branes whose positions are at N different points $Y' = 0, 2\pi R/N, \ldots, 2\pi R(N-1)/N$. Its boundary state is given by

$$|B_{\text{bulk}}, \epsilon\rangle_{\text{NSNS,RR}} = \sum_{a=0}^{N-1} |B_a, \epsilon\rangle_{\text{NSNS,RR}}\,, \tag{3.24}$$

where $|B_a, \epsilon\rangle_{\text{NSNS,RR}}$ is given by (3.22) (here we set $\tilde{y}'_0 = 0$). The explicit form of the boundary state is obtained in the same way as (3.17) and the result is

$$\begin{aligned} |B_{\text{bulk}}, \epsilon\rangle_{\text{NSNS,RR}} = {} & \frac{T_0/2}{2(2\pi R/N)} \times \\ & \times \sum_{\alpha\in\mathbb{Z}} e^{i\frac{Ny'}{R}\alpha} \left[|B, 0, +\rangle_{\text{NSNS,RR}} - (-1)^{k\alpha} |B, 0, -\rangle_{\text{NSNS,RR}}\right]. \end{aligned} \tag{3.25}$$

We can see that its boundary state is exactly the same form as that for a usual D-brane in type-II string (for even k) or a bound system of an electric and a magnetic D-brane away from each other with $\pi R/N$ in type-0 string (for odd k) on $\mathbb{S}^1$ with finite R/N (see figure 1). Note that this boundary state does not have the twisted sectors $l \neq 0$ but picks up only the untwisted sector $l = 0$ (or $n = N\alpha$ ($\alpha \in \mathbb{Z}$)). Thus

into each others completely except $R \to \infty$. Moreover this is consistent with the fact that there is only one massless RR-field in the Melvin model for the finite R, which can be seen in (2.23). One can see that another RR-field in type-0 theory is massive for the finite R, while it becomes massless in the uncompactified limit $R \to \infty$.

[28] Such a D0-brane, if it exists, should have a non-zero winding number $w \neq 0$ and violates the Dirichlet boundary condition (3.2) for Y'. Therefore we must consider the boundary state which breaks the U(1) current algebra symmetry.

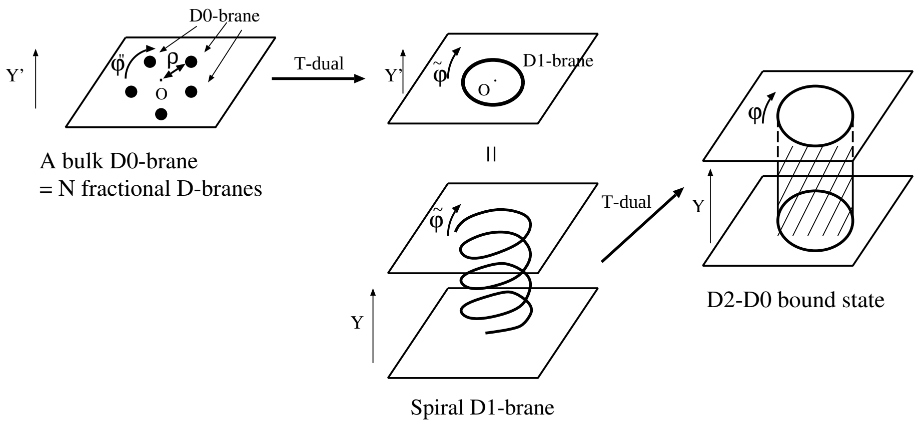

Figure 3: A bulk D0-brane can be regarded as a D2-D0 bound state in the Melvin background (2.2).

the bosonic zeromodes $x_0, \bar{x}_0$ indeed exist and the D0-brane can move around such that $\rho \neq 0$. To complete this argument one should also examine the first condition in eq. (3.2) carefully since the boundary state with $\rho \neq 0$ has a non-zero orbital angular momentum $\hat{J}_0$ in the zeromode part. Then the condition, which is equivalent to $(w - \frac{k}{N}\hat{J}_0)|B\rangle = 0$, is satisfied if $\hat{J}_0$ is a multiple of N. This requires that a bulk D-brane should consist of N fractional D-branes which are located at the N different points $X' = \rho\, e^{i(\theta + 2\pi k a/N)}$, $(a = 0, 1, \ldots, N-1)$ on the plane, where θ is an arbitrary constant (see figure 3). In this way we have shown that a bulk D0-brane exists if $\beta\alpha'/R \in \mathbb{Q}$ for any values of parameters[29] q and R.

Next we consider its vacuum amplitude. The result is N times as large as (3.11), while the second term of the open string hamiltonian (3.12) is modified as follows

$$\left(\frac{R\left(w - k\hat{J}\right)}{N\alpha'}\right)^2 \longrightarrow \begin{cases} \left(\frac{Rw}{N\alpha'}\right)^2, & \text{(for NS-sector and R-sector with even } k), \\ \left(\frac{R(w+1/2)}{N\alpha'}\right)^2, & \text{(for R-sector with odd } k). \end{cases}$$

This is because the angular momentum operator $\hat{J}$ takes integer (half-integer) values in NS-sector (R-sector) and we can erase the effect of $\hat{J}$ by the shift of w. From this hamiltonian we can see that there are no tachyonic modes though in the closed string sector there are in general. If we take the orbifold limit $R \to \infty$, the vacuum amplitude is N times as large as (3.21) without the orbifold projection $\sum_{l=0}^{N-1} g^l/N$, which is the same as that in the flat type-II theory for even k or type-0 theory for odd k.

[29] If the value of q is non zero, the vacuum amplitude with non-zero ρ includes an extra term which depends on q. This was recently shown in the case $\beta = 0$ $(N = 1)$ in the paper [68]. In the case of rational $\beta\alpha'/R = k/N$ the open string hamiltonian includes the extra contributions $\Delta H_O = \frac{\rho^2}{\pi^2\alpha'}\sin^2(\pi q R w/N + \pi k a/N)$. This deviation represents the twisted identification due to non-zero q and can be understood from the open string picture.

Finally we would also like to see how this D0-brane looks like in terms of the original Melvin coordinate $(X, \bar{X}, Y)$. Remember that the coordinates $(X', \bar{X}', Y')$ and $(X, \bar{X}, Y)$ are related with each other by performing the T-duality twice as $\varphi'' \to \tilde{\varphi} \to \varphi$ (see section 2.1). After the first T-duality procedure on φ'', the D0-brane located at $Y' =$ fixed, $\rho =$ fixed$(\neq 0)$ and $\varphi'' =$ fixed will be changed into a D1-brane wrapped on the circle $0 \leq \tilde{\varphi} < 2\pi\alpha'$. If we see this in terms of $(Y, \tilde{\varphi})$, the D1-brane can be viewed as a spiral D-string wrapped N times on the circle $0 \leq \tilde{\varphi} < 2\pi\alpha'$ and k times on the circle $0 \leq Y < 2\pi R$ because of the relation $Y = Y' - \beta\tilde{\varphi}$ in (2.6). If we take the second T-duality on $\tilde{\varphi}$, then we obtain a D2-brane wrapped on the torus $\mathbb{T}^2 \ni (Y, \varphi)$ (see figure 3). This D2-brane should be regarded as *a bound state of k D2-branes and N D0-branes* as can be seen from the winding numbers of the spiral D-string. Note that this is consistent with the asymptotic value of B-field in (2.2) $B_\varphi = -1/\beta = -N\alpha'/kR$ if $\beta\rho \gg 1$. From this value B_φ we can show that if we consider the world-volume theory of a D2-brane on a torus $\mathbb{T}^2$ in the Seiberg-Witten limit [39], we will obtain the theory on a non-commutative torus with the noncommutativity parameter $\theta = k/N$.

D1-branes Wrapped on $\mathbb{S}^1$. As we have seen, D0-branes in the Melvin background with the rational parameter are very similar to those in $\mathbb{Z}_N$ orbifolds. On the other hand, a D1-brane wrapped on $\mathbb{S}^1$ has an interesting structure which can not be explained clearly from the viewpoint of orbifold theories even though a D1-brane is formally transformed into a D0-brane by T-duality.

Here we consider the Melvin background with the rational parameter $qR = k/N$. The boundary state of a D1-brane can be constructed in the same way as that of a D0-brane. Then a single D1-brane is again pinned at the origin $\rho = 0$ (fixed point) in (2.2). However if we consider N D1-branes so that boundary state includes only the restricted winding sectors $w \in N\mathbb{Z}$, this bound system ("bulk" D1-brane) can move around in the plane $\mathbb{R}^2$ in the same way as in the previous case of D0-branes.

This behavior can be also explained geometrically as follows. Let us set $\beta = 0$ for simplicity and assume that a single D1-brane is placed at $\rho \neq 0$. Though this D1-brane obeys the Dirichlet boundary condition along $(X', \bar{X}') \in \mathbb{R}^2$, in the original coordinate $(X, \bar{X})$ it is rotated by the angle $2\pi k/N$ if it goes around $\mathbb{S}^1$ once, as shown in eq. (2.12). Thus it should wind N times around $\mathbb{S}^1$ in order to move around on the $\mathbb{R}^2$ plane (see figure 4). It is also useful to note that the geodesic lines along $\mathbb{S}^1$ are given by $\varphi + qy =$ constant from (2.3) and agrees with the world-volume of the D1-brane. This is a good tendency since it will minimize the mass of the D-brane.[30]

[30] For non-zero value of β the world volume of D1-brane seems to be the same $\varphi + qy =$ constant, which can be seen by examining the T-duality procedures as before. This is not exactly coincident with the geodesic line $\varphi + (q \pm \beta)y =$ constant for (2.2).

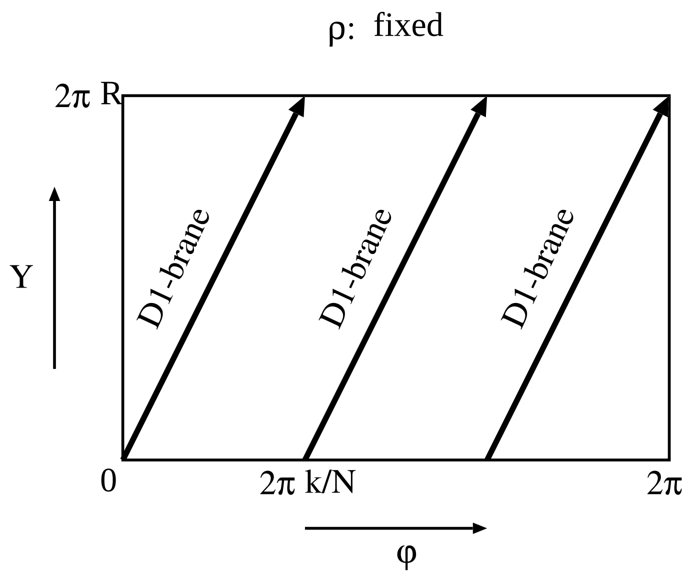

Figure 4: Spiral D1-branes

The analysis of the orbifold limit $R \to 0$ with $qR = k/N$ and $\beta\alpha'/R \to 0$ can be performed in the same way as before. A single D1-brane (below we reuse the coordinate $(X', \bar{X}', Y')$) which is sticked at the fixed point is identical to a fractional D0-brane in the limit by T-duality. On the other hand, the D1-brane winding N times around $\mathbb{S}^1$ corresponds to a bulk D0-brane which is made of N fractional D0-branes in the orbifold $\mathbb{C}/\mathbb{Z}_N$.

Let us consider the vacuum amplitude for a "bulk" D1-brane. The momentum part of the open string hamiltonian (the T-dual picture of (3.12)) is given by

$$\left(\frac{n - k\hat{J}}{NR}\right)^2 \longrightarrow \begin{cases} \left(\frac{n}{NR}\right)^2, & \text{(for NS-sector and R-sector with even k)}, \\ \left(\frac{n+1/2}{NR}\right)^2, & \text{(for R-sector with odd k)}. \end{cases}$$

Note that for odd k case the obtained D1-brane is really a system which consists of an electric "bulk" D1-brane and a magnetic "bulk" D1-brane. They wind N times around $\mathbb{S}^1$ and there is also a $\mathbb{Z}_2$ Wilson line on either D1-brane.

In this way we have found that a D1-brane wrapped on $\mathbb{S}^1$ in the Melvin background has the spiral structure (see figure 4). As can be seen from the open string hamiltonian there are again no tachyonic modes even though in the closed string sector there are tachyons in general.

Other Dp-branes. The analysis of other D-branes can be done in the same way. We have only to consider the three-dimensional Melvin geometry out of the ten-dimensional spacetime. Then both a D3-brane and a D2-brane which are extended in the $X', \bar{X}'$ directions are also allowed (see the table 2). Each of these becomes

Y'	$X', \bar{X}'$	Existence	Mobility	Tension	Form	Orbifold Limit
D	DD	all	Pinned	T_0	a D0-brane	fractional D0
D	DD	$\frac{\beta\alpha'}{R} = \frac{k}{N}$	Movable	NT_0	N D0-branes	bulk D0
D	DN	$\frac{\beta\alpha'}{R} = \frac{k}{N}$	Movable	NT_1	N D1-branes	bulk D1
D	NN	all	—	T_2	a D2-brane	"fractional" D2
N	DD	all	Pinned	T_1	a D1-brane	fractional D0
N	DD	$qR = \frac{k}{N}$	Movable	NT_1	N spiral D1-branes	bulk D0
N	DN	$qR = \frac{k}{N}$	Movable	NT_2	N spiral D2-branes	bulk D1
N	NN	all	—	T_3	a D3-brane	"fractional" D2

Table 2: D-brane spectrum in the Melvin background. Here we define the boundary condition of open strings in terms of free fields $(X', \bar{X}', Y')$. Note that for odd k we regard a Dp in the above as a bound state of an electric Dp and a magnetic Dp.

a "fractional"[31] D2-brane on the space $\mathbb{C}/\mathbb{Z}_N$ in the orbifold limit (for D3-branes we take a T-duality transformation in the Y' direction). This kind of D2-brane has the same tension as an ordinary D2-brane in almost the same way as in orbifold theories [63].

For the rational case there exists another D-brane which obeys the Neumann-Dirichlet or Dirichlet-Neumann boundary conditions for $X', \bar{X}'$ directions as we mentioned below (3.4). For bulk D-branes we can define such boundary conditions because on such D-branes the parameter γ which appears in the boundary states takes an integer value.[32] Especially note that these D-branes can not be defined as bound systems of N kinds of fractional D-branes such as (3.24).

Such an example also appears for $\beta\alpha'/R = k/N$ (or $qR = k/N$) with even N and odd k. In this background we can consider a D-brane with the untwisted sector $\hat{\gamma} = 0$ and *one twisted sector* $\hat{\gamma} = 1/2$, which is just the condition to allow the mixed boundary conditions (see below (3.4)).[33]

Here we have obtained the several D-branes in Melvin background, thus we summarize these results in the table 2.

3.3 Structure of D-branes for the irrational parameters

As we have already mentioned, the D-brane spectrum for the irrational parameters is remarkably different from the previous case of rational parameters. A single D-brane

[31] Usually in orbifold theories the Dp-brane ($p \geq 1$) is not called fractional because its tension is not divided by N, while in this paper "fractional" Dp-branes are ones which have the open string vacuum amplitude with the orbifold projection in the orbifold limit such as (3.21) in order to distinguish this from the "bulk" Dp-brane without the orbifold projection.

[32] Remember that bulk D-branes do not have twisted sectors.

[33] Of course we can define the D-brane with Neumann-Neumann or Dirichlet-Dirichlet boundary conditions for $X', \bar{X}'$ directions by the linear combination of $N/2$ kinds of fractional D-branes, while we can not construct the D-branes with mixed boundary conditions from the fractional D-branes.

can exist only at the origin $\rho = 0$ in (2.1) as before. However if one wants to put D-branes at the other points $\rho \neq 0$, then one must prepare infinitely many D-branes, which can be verified by calculating its tension in the same way as before. This fact matches the intuitive picture of the irrational case as a large-N limit of the orbifold $\mathbb{C}/\mathbb{Z}_N$ [13]. For example, let us consider a D1-brane wrapped on $\mathbb{S}^1$. In the irrational case the D1-brane along the geodesic line cannot return to the original point even if it goes around $\mathbb{S}^1$ any times. Thus the system will be a sort of a "foliation" of the cylinder $\rho =$ constant and may be like a 2-brane. Even though we cannot answer whether such a system can really exist, we can say that the D-brane spectrum for the irrational case is more restricted than that for the rational case.

3.4 World-volume theory

In the above we have constructed boundary states of various D-branes and have seen their geometric interpretations. Here we would like to briefly discuss the world-volume theory on these D-branes.

First note that the boundary states which have the Neumann (or Dirichlet) boundary condition in the Y' direction depend only on the parameter qR (or $\beta\alpha'/R$) and not on another. This fact is the reason why a D-brane in the background where *one of the parameters is rational* is treated as if it was a fractional D-brane, even though another parameter is irrational and this background is not equivalent to any freely acting orbifolds.

This fact will also lead to the intriguing property that the mass spectrum of open string which obeys the Dirichlet boundary condition along $\mathbb{S}^1$ can *show the Bose-Fermi degeneracy*[34] for $\beta = 0$ and $q \neq 0$ even if the *supersymmetry is completely broken* in the closed string sector. This mysterious phenomenon can be a relic of the supersymmetry in type-II theory, which is spontaneously broken [51] by the non-zero value of q. In other words the Bose-Fermi degeneracy only on the point-like D-brane along $\mathbb{S}^1$ implies the remained local supersymmetry, while the degeneracy does not occur on the D-brane wrapped on $\mathbb{S}^1$ because of the lack of global supersymmetry.

Next let us consider the relation to the quiver gauge theory [32]. In particular we take a D0-brane as an example. Let us remember the open string mass spectrum (3.23). If we take the orbifold limit $R \to \infty$, then it is easy to see that the $\mathbb{Z}_N$ projection of quiver theory

$$k\hat{J} - a + b = Nw \in N\mathbb{Z}\,, \tag{3.26}$$

appears, which restricts the spectrum to $g\ (= \exp[2\pi i \frac{k}{N}\hat{J}]) = \exp[2\pi i \frac{a-b}{N}]$. Thus the world-volume theory on D0-branes in the Melvin background can be regarded as a

[34] Note also that this fact may be correct only up to the one-loop order on the string coupling constant. This is because the closed one-loop (open two loop) amplitude between two D-branes contains the two point correlation function on the torus, and in this amplitude both parameters qR and $\beta\alpha'/R$ appear (see [13, 29]).

deformation of the quiver theory whose $\mathbb{Z}_N$ projection is softened. One can also see that the opposite limit $R \to 0$ leads to the mass spectrum of an ordinary D0-brane in type-II theory. Thus for a finite radius the world-volume theory is regarded as an interpolation between them. In particular the massless fields are the same as those of the quiver gauge theory of the orbifold $\mathbb{C}/\mathbb{Z}_N$.

4. D-branes in supersymmetric higher-dimensional models

The closed string backgrounds we have discussed above do not preserve any supersymmetry in general. If we would like to discuss D-brane charges, it will be more desirable to consider those in supersymmetric backgrounds. Therefore in this section we investigate D-branes in the higher-dimensional generalization of the Melvin model [13, 14], which includes the supersymmetric background. This model has a background of the form $\mathbb{M}_5 \times \mathbb{R}^{1,4}$, where $\mathbb{M}_5$ is a $\mathbb{S}^1 \ni Y$ fibration over $\mathbb{R}^2 \times \mathbb{R}^2 \ni (X^1, \bar{X}^1, X^2, \bar{X}^2)$. Correspondingly this model has four magnetic parameters q_1, q_2, β_1 and β_2, and the explicit metric of this model is given in the paper [13]. The four of eight (light-cone gauge) spinor fields in the Green-Schwarz formulation do not suffer from the phase factor when σ is shifted by 2π iff $q_1 = q_2, \beta_1 = \beta_2$ or $q_1 = -q_2, \beta_1 = -\beta_2$. Therefore we can conclude that in these specific cases half of thirty two supersymmetries will be preserved. Indeed we can also show that the partition function does vanish for these cases [13]. From the supergravity viewpoint, we can see this as follows. For simplicity, let us set $\beta_1 = \beta_2 = 0$. Then if the spinor fields go around the circle $\mathbb{S}^1$, they obtain the phase $e^{i\pi(\pm q_1 \pm q_2)R}$. Thus if $q_1 = q_2$ or $q_1 = -q_2$, there are sixteen Killing spinors. This special case $q_1 = \pm q_2, \beta_1 = \beta_2 = 0$ can be regarded as the "$9-11$" flip of the supersymmetric F5-brane [21]. We would also like to mention that similar arguments can also be generalized into much higher-dimensional backgrounds $\mathbb{M}_7$ (F3-brane), $\mathbb{M}_9$ (F1-brane) [13, 14]. The results on the behavior of D-branes discussed below will also be applied to these more generalized cases without any serious modifications.

4.1 Sigma model description and mass spectrum

If we consider the higher-dimensional generalizations of (2.4) its world-sheet action is given as follows:

$$\begin{aligned} S = \frac{1}{\pi\alpha'} \int d^2\sigma \, \big[& \bar{\partial}\rho\partial\rho + \bar{\partial}r\partial r + \rho^2 \bar{\partial}\check{\varphi}\partial\check{\varphi} + r^2 \bar{\partial}\check{\theta}\partial\check{\theta} + \\ & + \left(1 + \beta_1^2\rho^2 + \beta_2^2 r^2\right)^{-1} \left(\bar{\partial}Y + \beta_1\rho^2\bar{\partial}\check{\varphi} + \beta_2 r^2 \bar{\partial}\check{\theta}\right) \times \\ & \times \left(\partial Y - \beta_1\rho^2\partial\check{\varphi} - \beta_2 r^2 \partial\check{\theta}\right) \big] , \end{aligned} \tag{4.1}$$

where we have defined $\check{\varphi} = \varphi + q_1 Y$, $\check{\theta} = \theta + q_2 Y$. After T-duality of Y into $\tilde{Y}'$[35], we obtain

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \left[\bar{\partial}\rho\partial\rho + \bar{\partial}r\partial r + \bar{\partial}\tilde{Y}'\partial\tilde{Y}' + \rho^2\bar{\partial}\varphi''\partial\varphi'' + r^2\bar{\partial}\theta''\partial\theta'' \right], \tag{4.2}$$

where $\varphi'' \equiv \check{\varphi} - \beta_1\tilde{Y}'$, $\theta'' \equiv \check{\theta} - \beta_2\tilde{Y}'$ are the higher-dimensional generalization[36] of φ'' which appeared in eq. (2.10). The T-dual of $\tilde{Y}'$ is equivalent to Y' in eq. (2.7). The zero-mode of Y' is quantized in the same way as (2.20)

$$P'_L + P'_R = 2\left(\frac{n}{R} - q_1\hat{J}_1 - q_2\hat{J}_2\right), \qquad P'_L - P'_R = 2\left(\frac{Rw}{\alpha'} - \beta_1\hat{J}_1 - \beta_2\hat{J}_2\right), \tag{4.3}$$

where the angular momenta $\hat{J}_1 = \hat{J}_{1L} + \hat{J}_{1R}$ and $\hat{J}_2 = \hat{J}_{2L} + \hat{J}_{2R}$ are defined for each $\mathbb{R}^2$ as were done in eq. (2.15).

From (4.2) the bosonic fields $X^{1'} = \rho e^{i\varphi''}$, $X^{2'} = re^{i\theta''}$ and their superpartners $\psi_L^{1'}, \psi_R^{1'}, \psi_L^{2'}, \psi_R^{2'}$ become free fields and they obey the following boundary conditions for each $i = 1, 2$

$$\begin{aligned} X^{i'}(\tau, \sigma + 2\pi) &= e^{2\pi i\gamma_i} X^{i'}(\tau, \sigma), \\ \psi_L^{i'}(\tau, \sigma + 2\pi) &= e^{2\pi i\gamma_i}\psi_L^{i'}(\tau, \sigma), \qquad \psi_R^{i'}(\tau, \sigma + 2\pi) = e^{2\pi i\gamma_i}\psi_R^{i'}(\tau, \sigma), \\ \text{where} \qquad \hat{\gamma}_i &\equiv \gamma_i - [\gamma_i], \qquad \gamma_i \equiv q_i Rw + \beta_i\alpha'\left(\frac{n}{R} - q_1\hat{J}_1 - q_2\hat{J}_2\right). \end{aligned} \tag{4.4}$$

Then we can obtain the mass spectrum as follows

$$\begin{aligned} \frac{\alpha' M^2}{2} = &\frac{\alpha'}{2R^2}\left(n - q_1 R\hat{J}_1 - q_2 R\hat{J}_2\right)^2 + \frac{R^2}{2\alpha'}\left(w - \frac{\alpha'}{R}\beta_1\hat{J}_1 - \frac{\alpha'}{R}\beta_2\hat{J}_2\right)^2 + \\ &+ \hat{N}_R + \hat{N}_L - \sum_{i=1}^{2}\hat{\gamma}_i\left(\hat{J}_{Ri} - \hat{J}_{Li}\right), \end{aligned} \tag{4.5}$$

with the level matching constraint

$$\hat{N}_R - \hat{N}_L - nw + \sum_{i=1}^{2}[\gamma_i]\hat{J}_i = 0, \tag{4.6}$$

where $\hat{N}_{L,R}$ is defined in the same way as (B.5). From the above expression we can find the T-duality symmetry $q_i \leftrightarrow \beta_i$, $R \leftrightarrow \alpha'/R$ if $q_1\beta_2 = q_2\beta_1$. Note that the supersymmetric model ($q_1 = \pm q_2$, $\beta_1 = \pm\beta_2$) satisfies this condition.

It is also easy to see that if $n_1 + n_2$, $n_1 - n_2 \in 2\mathbb{Z}$, we obtain the following periodicity

$$Z(R, q_1, q_2, \beta_1, \beta_2) = Z\left(R, q_1 + \frac{n_1}{R}, q_2 + \frac{n_2}{R}, \beta_1, \beta_2\right), \tag{4.7}$$

and the periodicity for β_i can be also obtained by the T-duality.

[35] In this section we take the different process of T-duality from that in section 2.

[36] This correspondence can be shown if one notes that the relations (A.6) are rewritten as $-\partial\tilde{Y}' = \partial Y - \beta_1\rho^2\partial\varphi'' - \beta_2 r^2\partial\theta''$ and $\bar{\partial}\tilde{Y}' = \bar{\partial}Y + \beta_1\rho^2\bar{\partial}\varphi'' + \beta_2 r^2\bar{\partial}\theta''$.

4.2 Supersymmetric D-branes

We can consider D-branes in the higher-dimensional Melvin model in the same way as in section 3. The quantization of open strings can be performed because the action is already rewritten by free fields and we can construct the boundary states of D-branes. Its explicit form is almost the same as in eqs. (3.7) and (3.8). By using such boundary states we can obtain the vacuum amplitude, for example, for a D0-brane as follows

$$
\begin{aligned}
A = & \frac{\pi\alpha' V_0}{8\pi R}\int_0^\infty \frac{dt}{t^2}\times \\
& \times\Bigg[\Bigg\{\sum_{\gamma_1\in\mathbb{Z},\gamma_2\in\mathbb{Z}} \frac{|f_\gamma|^2}{(2\pi^2\alpha')^4}(\eta(it))^{-12}+ \\
& + \sum_{\gamma_1\notin\mathbb{Z},\gamma_2\in\mathbb{Z}} \frac{(-1)^{[\gamma_1]}|f_\gamma|^2}{(2\pi^2\alpha')^3}(\eta(it))^{-9}(\theta_1(\gamma_1|it))^{-1}+ \\
& + \sum_{\gamma_1\in\mathbb{Z},\gamma_2\notin\mathbb{Z}} \frac{(-1)^{[\gamma_2]}|f_\gamma|^2}{(2\pi^2\alpha')^3}(\eta(it))^{-9}(\theta_1(\gamma_2|it))^{-1}+ \\
& + \sum_{\gamma_1\notin\mathbb{Z},\gamma_2\notin\mathbb{Z}} \frac{(-1)^{[\gamma_1]+[\gamma_2]}|f_\gamma|^2}{(2\pi^2\alpha')^2}(\eta(it))^{-6}(\theta_1(\gamma_1|it))^{-1}(\theta_1(\gamma_2|it))^{-1}\Bigg\}\times \\
& \times\exp\left(-\frac{\pi\alpha'}{2R^2t}n^2\right)\left(\theta_1\left(\frac{\gamma_1+\gamma_2}{2}\middle|it\right)\right)^2\left(\theta_1\left(\frac{\gamma_1-\gamma_2}{2}\middle|it\right)\right)^2\Bigg], \quad (4.8)
\end{aligned}
$$

where $\gamma_i=\frac{\beta_i\alpha'}{R}n$, and the summations should be performed about $n\in\mathbb{Z}$ such that the conditions indicated below the symbol $\sum$ are satisfied. Note also that we have transformed the result obtained in the NS-R formulation into that in the Green-Schwarz formulation using the formula (C.4).

Such vacuum amplitude should be consistent with the Cardy's condition and this determines the coefficient f_γ which appears as in eq. (3.8). The result for a general Dp-brane is given by

$$
f_\gamma=\begin{cases}
\frac{1}{2}T_p\,, & (\gamma_1\in\mathbb{Z},\gamma_2\in\mathbb{Z})\,,\\
\frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma_1|}{2\pi^2\alpha'}\right)^{\mp1/2}T_p\,, & (\gamma_1\notin\mathbb{Z},\gamma_2\in\mathbb{Z})\,,\\
\frac{1}{\sqrt{2}}\left(\frac{|\sin\pi\gamma_2|}{2\pi^2\alpha'}\right)^{\mp1/2}T_p\,, & (\gamma_1\in\mathbb{Z},\gamma_2\notin\mathbb{Z})\,,\\
\left(\frac{|\sin\pi\gamma_1|}{2\pi^2\alpha'}\right)^{\mp1/2}\left(\frac{|\sin\pi\gamma_2|}{2\pi^2\alpha'}\right)^{\mp1/2}T_p\,, & (\gamma_1\notin\mathbb{Z},\gamma_2\notin\mathbb{Z})\,,
\end{cases} \quad (4.9)
$$

where the sign factor $\mp$ in the exponent of $|\sin\pi\gamma_i|$ $(i=1,2)$ takes $-$ for D-branes with Neumann-Neumann boundary condition, $+$ for D-branes with Dirichlet-Dirichlet boundary conditions for $X^{i'},\bar{X}^{i'}$ directions. In the Neumann-Dirichlet case one can define the boundary state in the same way only if γ is an integer or a half-integer. The above result (4.9) shows that a Dp-brane in the background has the ordinary tension T_p.

The structures of these D-branes are almost similar to those in the original Melvin background discussed in section 3. However we notice a remarkable property in this case: *under the condition that the supersymmetry in the closed string theory is preserved ($\beta_1 = \pm\beta_2$ and $q_1 = \pm q_2$) the open string one-loop amplitude (4.8) vanishes.* Thus the D-branes in this special background are stable BPS objects. In the orbifold limit[37] $R \to \infty$ with

$$\frac{\beta_1\alpha'}{R} = \pm\frac{\beta_2\alpha'}{R} = \frac{k}{N}\,, \qquad q_1 = q_2 = 0\,, \tag{4.10}$$

these D-branes are identified with BPS fractional D-branes [32, 33] in the supersymmetric ALE orbifolds $\mathbb{C}^2/\mathbb{Z}_N$. More generally, if $\beta_i\alpha'/R = k_i/N$ $(i = 1, 2)$ and if $k_1 + k_2$ is even, the D-brane in the limits becomes a fractional D-brane in type-II (not necessarily supersymmetric) orbifolds $\mathbb{C}^2/\mathbb{Z}_N$. On the other hand, if $k_1 + k_2$ is odd, it is divided into an electric and a magnetic fractional D-brane [35] in type-0 orbifolds $\mathbb{C}^2/\mathbb{Z}_N$.

According to the parameter values q_iR and $\beta_i\alpha'/R$ $(i = 1, 2)$ the D-brane spectrum dramatically changes again. For the irrational values only the D-branes which are stuck at the fixed point are allowed, while for the rational values there also exist the D-branes which can move around. The latter are made of N "fractional" D-branes, which can be seen as a generalization of a bulk D-brane in the orbifold theory $\mathbb{C}^2/\mathbb{Z}_N$ [32]. The detailed arguments of these D-branes are almost the same as before.

4.3 Comments on D-brane charges and K-theory

Before we conclude this paper, let us briefly mention the D-brane charges in the supersymmetric Melvin background. It is known that D-brane charges in type-IIB and -IIA theory are generally classified by K-theory $K^0(X)$ and $K^1(X)$ [44, 65], where X represents the manifold of the spacetime considered. First we assume that the parameters are given by (4.10). Then it is easy to see that the number of massless RR-fields are given by one for finite R and by N for the orbifold limit $R \to \infty$. In the orbifold language each of these N RR-fields comes from one untwisted sector and $N-1$ twisted sectors. Naively one may think that the number of types of Dp-branes should be given by N in accordance with the N-types of boundary states (3.22). However one should remember that for finite R the type of a fractional D-brane changes if it goes around the compactified circle. This shows that the types of fractional D-branes (twisted RR-charge) do not preserve unless we take the limit $R \to \infty$. Thus we conclude that the D-brane charge or equally the corresponding

[37] Equivalently one can take the limit $R \to 0$, $q_1R = \pm q_2R = k/N$, $\beta_1 = \beta_2 = 0$ by T-duality.

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K-group[38] should be given by the rather trivial result $K^0(M_5) = K^1(M_5) = \mathbb{Z}$ for finite R. On the other hand if we take the orbifold limit $R \to \infty$, then the K-group should be given by the equivalent K-theory $K^0_{\mathbb{Z}_N}(\mathbb{R}^4) = \mathbb{Z}^N$ [44, 65], which corresponds to the N-types of fractional D-branes on the orbifold $\mathbb{C}^2/\mathbb{Z}_N$. Thus the spectrum of D-brane charges does not appear to be continuously connected in the orbifold limit even though the masses of twisted RR-fields which couple D-branes continuously change. It would be interesting for the above facts to be understood from the viewpoint of the K-theory with H-flux like the twisted K-theory $K_H(X)$ discussed in [45, 46].

Next we would like to consider the case where $\beta_i\alpha'/R$ is irrational with $q_1 = q_2 = 0$. Then we find that in the limit $R \to \infty$ (a "large-N limit of the orbifold" [13]) there seems to be infinitely many massless RR-fields absorbing the zeromode along the circle $\mathbb{S}^1$. Thus we may have the "K-group" $K^0(M_5) = K^1(M_5) = \mathbb{Z}^\infty$ in this limit even though for the irrational parameters the space M_5 is no longer a smooth manifold nor a "good" singular manifold like ordinary orbifolds.

It may not be so nonsense even to ask whether one can explain D-brane charges in type-0 theory from the viewpoint of K-theory if one remembers that the type-II Melvin background approaches type-0 theories with the appropriate limits.

5. Conclusions and discussions

In this paper we have investigated various aspects of D-branes in Melvin backgrounds by employing the boundary state formalism. The boundary states are constructed in terms of the free field representations which can be obtained by applying T-duality to the original sigma model of the Melvin backgrounds. Then we can impose the ordinary Neumann or Dirichlet boundary conditions on the free fields, though the resulting D-branes are geometrically non trivial in the Melvin backgrounds.

The D-brane spectrum turns out to depend sensitively on whether the magnetic parameters qR and $\beta\alpha'/R$ are rational or irrational. In the former case the boundary states have the structure similar to orbifold theories and indeed they become the fractional D-branes[39] in the orbifolds $\mathbb{C}^n/\mathbb{Z}_N$ if we take the large radius (or small radius in T-dual picture) limits. For finite radius the N kinds of "fractional D-branes" which belong to the different irreducible representations of $\mathbb{Z}_N$ can be changed into each other by the monodromy if they go around the compactified circle. Even though a single ("fractional") D-brane cannot move away from the fixed point (or the origin), N different kinds of "fractional" D-branes can move around as a unit (a "bulk" D-

[38]The T-duality of the Melvin background is represented in terms of K-group as $K^0(M_5) = K^1(\hat{M}_5)$, where $\hat{M}_5$ is the T-dual of the five-dimensional Melvin background M_5.

[39]More precisely, we have explicitly constructed D-branes in the Melvin background which are reduced to fractional D0-branes (D1-branes) in one of the orbifold limits $R \to \infty$ ($R \to 0$) in section 3. We left the discussion on the other limit as a future problem.

brane). On the other hand, in the case of irrational parameters one needs the infinite number of the "fractional" D-branes in order to move away from the fixed point. We left the possibility of the existence of such D-branes as an unresolved problem.

The orbifold theory we obtain in the limits is defined in either type-0 or type-II string theory depending on the rational parameters. In the type-0 case we found that a D-brane behaves like a combined system of an electric fractional D-brane and a magnetic fractional D-brane.

The individual effects of two parameters q and β are roughly as follows. The non-zero value of the parameter q twists the plane $\mathbb{R}^2$ spirally along the circle $\mathbb{S}^1$. As a result the world-volume of a D-brane which wraps the circle is bended along a spiral geodesic line (see eq. (2.3)). The effect of non-zero β, which means the non-zero H-flux, is more complicated because one should mix the free fields with its T-dualized ones as in eq. (2.6) and (2.12) in order to return back to the original Melvin sigma model (2.4). This complexity may be due to the presence of H-flux. Owing to this T-duality transformation, a bulk D0-brane at $\rho \neq 0$ in terms of the free fields $(X', \bar{X}', Y')$ is interpreted as a D2-D0 bound state which wraps the two-dimensional torus ($\rho =$ fixed) in the original Melvin background (2.2). Interestingly, this two-dimensional torus can be regarded as a non-commutative torus $\mathcal{A}_\theta$ with the noncommutativity $\theta = \beta\alpha'/R$ if we assume that the B-field is large $\beta\rho \gg 1$ and apply the argument in [39]. The observed dependence of D-branes on the parameters can be now reinterpreted as that on the parameter of the non-commutative torus $\mathcal{A}_\theta$. The corresponding operator algebra K-theory $K(\mathcal{A}_\theta) = \mathbb{Z} + \theta\mathbb{Z}$ [66] shows the analogous difference between the rational and the irrational case, where the value $\mathbb{Z} + \theta\mathbb{Z} (\in \mathbb{R})$ means the dimension of the corresponding projective module. Indeed two $\mathbb{Z}$ charges in the K-group $K(\mathcal{A}_\theta)$ represent the D2-brane and the D0-brane charges as is clearly explained from the viewpoint of the tachyon condensation in the open string theory [67]. It would be interesting to examine the non-commutative algebra on such D-branes by using the free field calculations since this background possesses non-zero H-flux (for a general discussion on the relation between H-flux and non-commutative geometry see [40]).

Such an interpretation of a D0-brane in the free field model (2.7) as a D2-D0 bound state in the Melvin sigma model (2.4) seems to raise some intriguing questions. The first question is whether an ordinary D0-brane can also exist in the Melvin background (2.2) and what is the form of its boundary state if it exists. The second is why such a D2-D0 bound state can exist in the Melvin background which does not wrap any homologically non-trivial cycle. We leave these issues as future problems.

It is also an interesting result that the boundary states of D-branes depend only on either of two parameters. This phenomenon leads to the strange Bose-Fermi degeneracy on specific D-branes even in non-supersymmetric string backgrounds. We pointed out the interpretation of this as a remnant of the spontaneously broken supersymmetry.

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Finally we would like to note that we can apply the above results for the original Melvin background (2.2) to its higher-dimensional generalizations without any serious modifications as we have seen in section 4. For specific values of parameters the model preserves some supersymmetries and the D-branes there become BPS states. Thus it would be interesting to explore their supersymmetric world-volume theories and their statuses from the viewpoint of the string dualities.

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Note Added. After completing this paper we noticed the paper [68] on the net, which has some overlaps with our paper.

A. T-duality for Kaluza-Klein background

The coordinate of world-sheet is $z = \sigma_1 + i\sigma_2$ and we define its partial derivative as $\partial = \frac{1}{2}(\partial_1 - i\partial_2), \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. Let us first consider the following bosonic sigma model:

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \left[(G_{ij}(X) + B_{ij}(X))\partial X^i \bar{\partial} X^j + \frac{1}{4}\alpha' R^{(2)}\phi(X) \right]. \tag{A.1}$$

After substituting the Kaluza-Klein background like (2.2) we obtain

$$\begin{aligned} S = \frac{1}{\pi\alpha'} \int d^2\sigma \Big[&(G_{\mu\nu}(X) + B_{\mu\nu}(X))\bar{\partial} X^\nu \partial X^\mu + \\ &+ e^{2\sigma(X)} \left(\bar{\partial} Y + A_\mu \bar{\partial} X^\mu\right) \left(\partial Y + A_\mu \partial X^\mu\right) + \\ &+ B_\mu \left(\bar{\partial} Y \partial X^\mu - \bar{\partial} X^\mu \partial Y\right) + \frac{1}{4}\alpha' R^{(2)}\phi(X) \Big]. \end{aligned} \tag{A.2}$$

Then we can perform T-duality along Y direction ($\mathbb{S}^1$ with radius R) since the fields $G_{\mu\nu}(X), B_{\mu\nu}(X), \phi(X)$ do not depend on Y [50]. Introducing the auxiliary vector field $V, \bar{V}$ we can rewrite (A.2) as follows (we show only non-trivial parts)

$$\begin{aligned} S = \frac{1}{\pi\alpha'} \int d^2\sigma \Big[&e^{2\sigma(X)} \left(\bar{V} + A_\mu \bar{\partial} X^\mu\right) \left(V + A_\mu \partial X^\mu\right) + B_\mu \left(\bar{V} \partial X^\mu - \bar{\partial} X^\mu V\right) + \\ &+ \left(\bar{V} \partial \tilde{Y} - \bar{\partial} \tilde{Y} V\right) \Big], \end{aligned} \tag{A.3}$$

where the new field $\tilde{Y}$ is compactified on a circle with the radius α'/R. If we first integrate $\tilde{Y}$, then we obtain $\bar{\partial} V - \partial \bar{V} = 0$ and the vector field $V, \bar{V}$ can be written as $V = \partial Y,\ \bar{V} = \bar{\partial} Y$. Indeed one can easily see that this field Y has the periodicity

$Y \sim Y + 2\pi R$ as expected.[40] Next it is straightforward to integrate out V first and one obtains

$$
\begin{aligned}
S = \frac{1}{\pi\alpha'} \int d^2\sigma \Big[\cdots + e^{-2\sigma(X)} \left(\bar{\partial}\tilde{Y} + B_\mu \bar{\partial} X^\mu\right) \left(\partial\tilde{Y} + B_\mu \partial X^\mu\right) + \\
+ A_\mu \left(\bar{\partial}\tilde{Y}\partial X^\mu - \bar{\partial} X^\mu \partial\tilde{Y}\right) + \frac{1}{4}\alpha' R^{(2)}(\phi(X) - \sigma(X))\Big], \quad \text{(A.4)}
\end{aligned}
$$

where the shift of the dilaton field can be determined from the condition of the conformal invariance (vanishing beta-function) [50]. Thus we have obtained the following T-duality transformation:

$$
\begin{aligned}
&\sigma'(X) = -\sigma(X)\,, \qquad A'_\mu(X) = B_\mu(X)\,, \qquad B'_\mu(X) = A_\mu(X)\,, \\
&\phi'(X) = \phi(X) - \sigma(X)\,. \qquad \text{(A.5)}
\end{aligned}
$$

Note also that the equation of motion of (A.2) is equivalent to that of (A.4) via the rule:

$$
\begin{aligned}
\partial\tilde{Y} &= -B_\mu \partial X^\mu - e^{2\sigma}(\partial Y + A_\mu \partial X^\mu)\,, \\
\bar{\partial}\tilde{Y} &= -B_\mu \bar{\partial} X^\mu + e^{2\sigma}(\bar{\partial} Y + A_\mu \bar{\partial} X^\mu)\,. \qquad \text{(A.6)}
\end{aligned}
$$

In this paper we discuss superstring models and therefore we need the supersymmetric generalization of the above arguments. The simplest way to do this is to use the $N = 1$ superspace formalism. One has only to replace ∂ and $\bar{\partial}$ with the super covariant derivatives $D_\theta = \partial_\theta + \theta\partial$ and $D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial}$ and replace the bosonic vector field V with the fermionic vector super field $\mathbf{W}$. The bosonic scalar field $X(z, \bar{z})$ should also be changed into $\mathbf{X}(z, \bar{z}) = X(z, \bar{z}) + i\theta\psi'_L(z) + i\bar{\theta}\psi'_R(\bar{z}) + \cdots$. The calculations are almost the same as before. For example, eq. (A.6) is replaced with

$$
\begin{aligned}
D_\theta \tilde{\mathbf{Y}} &= -B_\mu D_\theta \mathbf{X}^\mu - e^{2\sigma}(D_\theta \mathbf{Y} + A_\mu D_\theta \mathbf{X}^\mu)\,, \\
D_{\bar{\theta}} \tilde{\mathbf{Y}} &= -B_\mu D_{\bar{\theta}} \mathbf{X}^\mu + e^{2\sigma}(D_{\bar{\theta}} \mathbf{Y} + A_\mu D_{\bar{\theta}} \mathbf{X}^\mu)\,. \qquad \text{(A.7)}
\end{aligned}
$$

Let us see a useful example: T-duality of the background (2.2). Utilizing the above arguments we can transform q into β by T-duality. This fact is reconfirmed in the mass spectrum (2.23).

[40] To see this, one should note the relation $d\tilde{Y}$ =(closed form)+(non-trivial cohomology). The non-trivial part is discretized due to the periodicity $\tilde{Y} \sim \tilde{Y} + 2\pi\frac{\alpha'}{R}$. Then the term $\sim \int d\tilde{Y} \wedge V$ in (A.3) leads to the weight $\exp(i\frac{\tilde{w}}{R}\int_C V)$, where $\tilde{w}$ is the winding number of the field $\tilde{Y}$ and C is a one-cycle of the world-sheet. Since one should take a summation over w, the integration is quantized as $\int_C V \in 2\pi R\mathbb{Z}$. This shows that the period of Y is $2\pi R$.

B. Mode expansion

As we have seen in section 2, the theory can be represented by the free bosonic fields $X', \bar{X}',\ Y'$ and their superpartners $\psi'_{L,R}, \bar{\psi}'_{L,R},\ \eta'_{L,R}$ with the boundary conditions (2.18), (2.20) and (2.21). Thus we can find the following mode expansions

$$\begin{aligned}
X'(z,\bar{z}) &= X'_L(z) + X'_R(\bar{z}) \\
&= i\sqrt{\alpha'}\sum_m \frac{1}{m-\gamma}\alpha_{m-\gamma}z^{-m+\gamma} + i\sqrt{\alpha'}\sum_m \frac{1}{m+\gamma}\tilde{\alpha}_{n+\gamma}\bar{z}^{-m-\gamma}\,, \\
\bar{X}'(z,\bar{z}) &= \bar{X}'_L(z) + \bar{X}'_R(\bar{z}) \\
&= i\sqrt{\alpha'}\sum_m \frac{1}{m+\gamma}\bar{\alpha}_{m+\gamma}z^{-m-\gamma} + i\sqrt{\alpha'}\sum_m \frac{1}{m-\gamma}\bar{\tilde{\alpha}}_{m-\gamma}\bar{z}^{-m+\gamma}\,, \\
\psi'_L(z) &= \sqrt{\alpha'}\sum_r \psi_{r-\gamma}z^{-r+\gamma-\frac{1}{2}}\,, \qquad \psi'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \tilde{\psi}_{r+\gamma}\bar{z}^{-r-\gamma-\frac{1}{2}}\,, \\
\bar{\psi}'_L(z) &= \sqrt{\alpha'}\sum_r \bar{\psi}_{r+\gamma}z^{-r-\gamma-\frac{1}{2}}\,, \qquad \bar{\psi}'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \bar{\tilde{\psi}}_{r-\gamma}\bar{z}^{-r+\gamma-\frac{1}{2}}\,, \\
Y'(z,\bar{z}) &= y' - i\frac{\alpha'}{2}P'_L\ln z - i\frac{\alpha'}{2}P'_R\ln\bar{z} + \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\beta_m z^{-m} + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\tilde{\beta}_m\bar{z}^{-m}\,, \\
\eta'_L(z) &= \sqrt{\alpha'}\sum_r \eta_r z^{-r-\frac{1}{2}}\,, \qquad \eta'_R(\bar{z}) = \sqrt{\alpha'}\sum_r \tilde{\eta}_r\bar{z}^{-r-\frac{1}{2}}\,, \qquad \text{(B.1)}
\end{aligned}$$

and (anti)commutation rules

$$\begin{aligned}
[\alpha_{m-\gamma}, \bar{\alpha}_{n+\gamma}] &= (m-\gamma)\delta_{m,-n}\,, \qquad [\tilde{\alpha}_{m+\gamma}, \bar{\tilde{\alpha}}_{n-\gamma}] = (m+\gamma)\delta_{m,-n}\,, \\
\{\psi_{r-\gamma}, \bar{\psi}_{s+\gamma}\} &= \delta_{r,-s}\,, \qquad \left\{\tilde{\psi}_{r+\gamma}, \bar{\tilde{\psi}}_{s-\gamma}\right\} = \delta_{r,-s}\,, \\
[\beta_m, \beta_n] &= m\delta_{m,-n}\,, \qquad \left[\tilde{\beta}_m, \tilde{\beta}_n\right] = m\delta_{m,-n}\,, \\
\{\eta_r, \eta_s\} &= \delta_{r,-s}\,, \qquad \{\tilde{\eta}_r, \tilde{\eta}_s\} = \delta_{r,-s}\,. \qquad \text{(B.2)}
\end{aligned}$$

The $\mathcal{N}=1$ super-Virasoro generators $L_m, \tilde{L}_m, G_r, \tilde{G}_r$ are obtained in the usual way because the action is written by free fields

$$\begin{aligned}
L_m &= :\frac{1}{2}\sum_k \beta_{m-k}\beta_k + \sum_k \bar{\alpha}_{m-k+\gamma}\alpha_{k-\gamma} + \\
&\quad + \frac{1}{2}\sum_r \left(r - \frac{m}{2}\right)\eta_{m-r}\eta_r + \sum_r \left(r - \gamma - \frac{m}{2}\right)\bar{\psi}_{m-r+\gamma}\psi_{r-\gamma} :\,, \\
G_r &= \sum_k \left(\beta_k\eta_{r-k} + \alpha_{k-\gamma}\bar{\psi}_{r-k+\gamma} + \bar{\alpha}_{k+\gamma}\psi_{r-k+\gamma}\right), \qquad \text{(B.3)}
\end{aligned}$$

where :∼: is the conformal normal ordering. For example for $0 < \gamma < 1/2$, L_0 for NS-sector becomes[41]

$$L_0 = \frac{\alpha'}{4} P_L'^2 + \sum_{k=1}^{\infty} \beta_{-k}\beta_k + \sum_{k=1}^{\infty} \bar{\alpha}_{-k+\gamma}\alpha_{k-\gamma} + \sum_{k=0}^{\infty} \alpha_{-k-\gamma}\bar{\alpha}_{k+\gamma} + \\ + \sum_{r=1/2}^{\infty} r\eta_{-r}\eta_r + \sum_{r=1/2}^{\infty} (r-\gamma)\bar{\psi}_{-r+\gamma}\psi_{r-\gamma} + \sum_{r=1/2}^{\infty} (r+\gamma)\psi_{-r-\gamma}\bar{\psi}_{r+\gamma} + \frac{\gamma}{2}, \quad \text{(B.4)}$$

where P_L' is given by (2.20). Note that here we abbreviate the contributions from other directions than Y', X' and $\bar{X}'$. L_0 for R-sector is almost the same except that r runs integer values and that the zero point energy shifts. The antiholomorphic components $\tilde{L}_m, \tilde{G}_r$ is written in the same form.

For the later use, we define the operators $\hat{N}_L, \hat{N}_R$ (we show only the result in NSNS sector)

$$\hat{N}_L = \sum_{k=1}^{\infty} \frac{k}{k-\gamma}\bar{\alpha}_{-k+\gamma}\alpha_{k-\gamma} + \sum_{k=1}^{\infty} \frac{k}{k+\gamma}\alpha_{-k-\gamma}\bar{\alpha}_{k+\gamma} + \sum_{k=1}^{\infty} \beta_{-k}\beta_k + \\ + \sum_{r=1/2}^{\infty} r\bar{\psi}_{-r+\gamma}\psi_{r-\gamma} + \sum_{r=1/2}^{\infty} r\psi_{-r-\gamma}\bar{\psi}_{r+\gamma} - \frac{1}{2},$$

$$\hat{N}_R = \sum_{k=1}^{\infty} \frac{k}{k-\gamma}\tilde{\alpha}_{-k+\gamma}\bar{\tilde{\alpha}}_{k-\gamma} + \sum_{k=1}^{\infty} \frac{k}{k+\gamma}\bar{\tilde{\alpha}}_{-k-\gamma}\alpha_{k+\gamma} + \sum_{k=1}^{\infty} \tilde{\beta}_{-k}\tilde{\beta}_k + \\ + \sum_{r=1/2}^{\infty} r\tilde{\psi}_{-r+\gamma}\bar{\tilde{\psi}}_{r-\gamma} + \sum_{r=1/2}^{\infty} r\bar{\tilde{\psi}}_{-r-\gamma}\tilde{\psi}_{r+\gamma} - \frac{1}{2}. \quad \text{(B.5)}$$

The last constant term $-1/2$ is replaced by 0 for RR-sector.

The angular momentum operators $\hat{J}_L, \hat{J}_R$, are also defined to be (we show the result for NSNS-sector with $0 < \gamma < 1/2$)

$$\hat{J}_L = -\sum_{k=1}^{\infty} \frac{1}{k-\gamma}\bar{\alpha}_{-k+\gamma}\alpha_{k-\gamma} + \sum_{k=0}^{\infty} \frac{1}{k+\gamma}\alpha_{-k-\gamma}\bar{\alpha}_{k+\gamma} - \\ - \sum_{r=1/2}^{\infty} \bar{\psi}_{-r+\gamma}\psi_{r-\gamma} + \sum_{r=1/2}^{\infty} \psi_{-r-\gamma}\bar{\psi}_{r+\gamma} + \frac{1}{2},$$

$$\hat{J}_R = \sum_{k=1}^{\infty} \frac{1}{k-\gamma}\tilde{\alpha}_{-k+\gamma}\bar{\tilde{\alpha}}_{k-\gamma} - \sum_{k=0}^{\infty} \frac{1}{k+\gamma}\bar{\tilde{\alpha}}_{-k-\gamma}\alpha_{k+\gamma} + \\ + \sum_{r=1/2}^{\infty} \tilde{\psi}_{-r+\gamma}\bar{\tilde{\psi}}_{r-\gamma} - \sum_{r=1/2}^{\infty} \bar{\tilde{\psi}}_{-r-\gamma}\tilde{\psi}_{r+\gamma} - \frac{1}{2}. \quad \text{(B.6)}$$

The last constant term for RR-sector is the same as the above.

[41]Note that if γ is out of this region, the above L_0 is not positive definite, therefore we have to redefine the ground state correctly.

C. Formulae of θ-functions

Here we summarize the formulae of θ-functions. First define the following θ-functions:

$$\begin{aligned}
\eta(\tau) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,,\\
\theta_1(\nu|\tau) &= 2q^{1/8}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1-e^{2i\pi\nu}q^n\right)\left(1-e^{-2i\pi\nu}q^n\right),\\
\theta_2(\nu|\tau) &= 2q^{1/8}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1+e^{2i\pi\nu}q^n\right)\left(1+e^{-2i\pi\nu}q^n\right),\\
\theta_3(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1+e^{2i\pi\nu}q^{n-\frac{1}{2}}\right)\left(1+e^{-2i\pi\nu}q^{n-\frac{1}{2}}\right),\\
\theta_4(\nu|\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1-e^{2i\pi\nu}q^{n-\frac{1}{2}}\right)\left(1-e^{-2i\pi\nu}q^{n-\frac{1}{2}}\right),
\end{aligned}\tag{C.1}$$

where we have defined $q=e^{2i\pi\tau}$.

Next we show the modular properties as follows

$$\begin{aligned}
\eta(\tau) &= (-i\tau)^{-1/2}\eta\left(-\frac{1}{\tau}\right),\\
\theta_1(\nu|\tau) &= i(-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_1\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right),\\
\theta_2(\nu|\tau) &= (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_4\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right),\\
\theta_3(\nu|\tau) &= (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_3\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right),\\
\theta_4(\nu|\tau) &= (-i\tau)^{-1/2}e^{-\pi i\frac{\nu^2}{\tau}}\theta_2\left(\frac{\nu}{\tau}\middle|-\frac{1}{\tau}\right).
\end{aligned}\tag{C.2}$$

Their quasi periodicities are also given by

$$\begin{aligned}
\theta_1(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_1(\nu|\tau)\,,\\
\theta_2(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_2(\nu|\tau)\,,\\
\theta_3(\nu+\tau|\tau) &= e^{-2\pi i\nu-\pi i\tau}\theta_3(\nu|\tau)\,,\\
\theta_4(\nu+\tau|\tau) &= -e^{-2\pi i\nu-\pi i\tau}\theta_4(\nu|\tau)\,.
\end{aligned}\tag{C.3}$$

It is useful to note the relation

$$\prod_{a=1}^{4}\theta_3(\nu_a|\tau)-\prod_{a=1}^{4}\theta_2(\nu_a|\tau)-\prod_{a=1}^{4}\theta_4(\nu_a|\tau)+\prod_{a=1}^{4}\theta_1(\nu_a|\tau)=2\prod_{a=1}^{4}\theta_1(\nu'_a|\tau)\,,\tag{C.4}$$

where we have defined

$$
\begin{aligned}
2\nu_1' &= \nu_1 + \nu_2 + \nu_3 + \nu_4\,, \qquad & 2\nu_2' &= \nu_1 + \nu_2 - \nu_3 - \nu_4\,, \\
2\nu_3' &= \nu_1 - \nu_2 + \nu_3 - \nu_4\,, \qquad & 2\nu_4' &= \nu_1 - \nu_2 - \nu_3 + \nu_4\,.
\end{aligned}
\tag{C.5}
$$

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